

A
Treatise on Arithmetic



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A
TREATISE
ON
ARITHMETIC.



BOOK I.
WHOLE NUMBERS.
CHAPTER I.

ON THE IDEAS OF NUMBER, AND THEIR EXPRESSION
BY WORDS.

(1.) THE early period of life at which we begin to acquire ideas of number, and the influence produced on our minds by becoming familiar with the language of numeration before any notions of the higher classes of number are formed, throw great difficulties in the way of any attempt to retrace the steps by which the art of counting arrived at its present high state of perfection. Under these circumstances we must naturally direct our enquiries to the state of arithmetic among people less advanced in civilisation than ourselves. But it is a remarkable fact, that while other sciences are almost unknown beyond the limits of civilisation, there is no example of a people without a system of numeration more or less extensive and perfect, with the exception of a few savage tribes, whose notions of number are singularly limited.

It is, however, apparent, that before a people could make any progress in numeration beyond a few of the smaller combinations which may be signified by the fingers, two things were indispensably necessary to be accomplished.

1. To devise a method by which the mind should form a clear and distinct notion of the number of individual objects in any assemblage, however great might be the multitude of which the assemblage consisted.

2. To devise a system of names for numbers, such ~~that~~ it might be possible for the memory to retain the terms necessary to express the unlimited variety of notions which any system of numeration must necessarily involve.

(2.) It is, perhaps, one of the most curious and interesting facts in the history of the human mind, that all the nations of the earth, which have possessed any system of numeration, have adopted methods of solving these two problems, which, in their general features, and in all essential points, are identical, and which, in some cases, are the same, even in their most minute details.

Let us suppose two boxes, containing a large number of individual objects of the same kind, such as counters, placed before a person unacquainted with the received nomenclature of number, and who wishes to form a notion of the number of counters in each box, so as to be able to compare them, and to pronounce which collection is the greater. Such a person would, probably begin by withdrawing the counters one by one from one of the boxes, disposing them side by side, so that, upon inspection, he would have a distinct notion of the number withdrawn. He would, however, presently find that the collection withdrawn would become so numerous, that a mere inspection would give him no clear or distinct notion of it. In fact, he would feel himself in that situation that having withdrawn a collection from each box, and having each spread before him he would not immediately perceive which were the more numerous.

Instead, therefore, of spreading before him a large number of the counters, he will, in the first place, collect into one group only so many as he can form a clear idea of by inspection. Let us suppose that he fixes upon the number *six*; he then withdraws six more, and

disposes them in another group, placed beside the first. He proceeds in the same manner to form a third group of six, and continues to form groups of six, by constantly withdrawing the counters from one of the boxes. When six of these groups are formed, it is obvious that he will have as clear a conception of the number of counters which they contain as he has of the number of counters contained in any one of the groups; but for the same reason that he did not place more than six counters in each group in the first instance he will not now collect together more than six groups. Nothing, however, prevents him from regarding these six groups as a single group of a higher order, and pursuing his former method until he form a collection of six groups more; he will then have two collections of six groups, six counters being contained in each group. In the same manner he may proceed to form the third collection of six groups, and so on, until he has formed six such collections. It will be obvious upon the slightest reflection that his notion of the number of counters contained in the six collections thus formed is as clear and definite as is his notion of only six counters. If a proof of this be required, it may be readily given. Let him form six similar collections of six groups from the other box: but let us suppose that one of the groups in one collection shall be deficient by a single counter: he will then be immediately conscious that the number of counters withdrawn from one box exceeds the number withdrawn from the other by a single counter, and his idea of the number of counters withdrawn from the one box is as distinct from his idea of the number withdrawn from the other box as his notion of a single counter would be distinct from his notion of the total number withdrawn.

We may imagine the first group of six counters withdrawn to be thus arranged:—

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

The number of objects here is so limited that we have an immediate and complete perception of it the moment we look at it. If but a single object be added or taken away, we become immediately conscious that the number is increased or diminished. The first collection of six groups withdrawn may be represented thus : —

0 0 0	0 0 0	0 0 0
0 0 0	0 0 0	0 0 0
0 0 0	0 0 0	0 0 0
0 0 0	0 0 0	0 0 0

Each group in this collection is similar to each counter in a single group, and a single group withdrawn or added would become immediately apparent, as would a single counter deficient or in excess in any one of the groups. It will be equally easy to imagine six such collections of groups arranged in the same manner.

The spirit of this method consists in forming, in the first instance, a group of objects so limited in number that the mind can picture to itself so clear and distinct a notion of them that the increase or diminution of their number by one would be immediately perceptible ; these groups are then treated as the individual objects were treated in the first group, being formed into collections, having the same number of groups as there are individual objects in each group. These collections of groups are again collected in assemblages of a higher order, consisting still of the same number of collections as there were individuals in the first group, and this process has obviously no limit.

— (3.) Such are the general features of the method of numeration in which all nations of the earth without a single exception have concurred. The only point in the system which is arbitrary is the number of individuals which may be adopted to form the first group, and we accordingly find that systems of numeration differ in this particular ; still even here there is a surprising coincidence, even among people so far removed as to preclude all possibility of conventional arrange-

ment. The fingers were naturally the first objects which presented to the mind the idea of number; and they furnished also a set of natural counters by which the number of other things might be marked and expressed. The fingers being continually in view familiarised the mind with the contemplation of every number of objects not exceeding ten. It was natural, therefore, that *ten* should be adopted as the number of objects to form the first group. In the system of numeration, which has been just explained, and in the example which we have already given, we have adopted *six*; had *ten* been adopted as the *radix* of our example, we should have first formed a group of ten counters, and then formed in like manner nine other groups. These ten groups would then form a group of a higher order, in which each group would play the same part as the individual counters did in the original groups; and we should have proceeded in the same manner to form ten assemblages of the next order, and so on.

A curious example, illustrative of the universality of the method of forming ideas of large numbers, which has been above explained, is afforded in the history of the island of Madagascar.* When the people of that island wish to count a great multitude of objects, such, for example, as the number of men in a large army, they cause the objects to pass in succession through a narrow passage before those whose business it is to count them. For each object that passes they lay down a stone in a certain place; when all the objects to be counted have passed, they then dispose the stones in heaps of ten; they next dispose these heaps in groups, having ten heaps each, so as to form hundreds; and in the same way would dispose the groups of hundreds so as to form thousands, until the number of stones has been exhausted.

Although *ten* has been so generally adopted as the *radix* of systems of numeration as to leave no doubt of

* Histoire de la Grande Isle de Madagascar, par de Flacourt, ch. xxviii. 1661, quoted by Peacock

its origin, yet it is not the only one which has been used, nor is it the only radix having a natural origin. The fingers of one hand rendered the number five familiar to the mind before the conception of ten as a distinct number presented itself. It was even more natural and obvious that the fingers should be contemplated as two groups of five than as a single group of ten. We accordingly find, in some instances, the number five taken as the radix of numeration, but there are inconveniences arising from its smallness which will be more clearly perceived hereafter. The existence of the members and principal organs of the human body, and of the bodies of other animals in pairs, furnished a natural foundation for adopting *two* as a *radix*; but the process of grouping any considerable number by pairs, and by pairs only, would be attended with still more inconvenience than the quinary radix.

(4.) The solution of the second problem above stated by the discovery of an appropriate system of names to express the numbers, of which clear ideas may be formed by the method of grouping and arrangement just explained, must needs have proceeded, *pari passu*, with the solution of the first. Distinct names would be given to every collection of objects not exceeding the number selected to form the first group. The same succession of names, properly modified, would express all collections of groups less in number than the radix: thus, in the first example above given, we should require six distinct names to express all numbers not exceeding the number six, which was selected as the radix of the system. The same names, however, which expressed the number of individual objects as far as *sfx* would likewise express the number of groups as far as six. A new name would here become necessary to distinguish six groups from six individuals. By properly combining the twelve names, thus formed, we should be enabled to express all numbers under six collections of six groups; but here a new term must be introduced. Such are the general features of the

method of naming numbers, which has been adopted in all languages.

(5.) What we have here stated will be more clearly apprehended when illustrated by the system of decimal numeration. *Ten* being the radix in that system, the numbers, as far as *ten* inclusive, have distinct and independent names, *one, two, three, four, five, six, seven, eight, nine, ten*. The names *eleven* and *twelve* being anomalous, we shall, for the present, substitute for them *ten-one* and *ten-two*. The names, then, for all numbers expressing the first group, and part of the second, would be *ten-one, ten-two, ten-three, ten-four, ten-five, ten-six, ten-seven, ten-eight, ten-nine*. Two groups of ten are expressed by *two-tens*, which abridged, is *twenty*. All collections of individuals consisting of two groups and a part of a third are expressed by *twenty-one, twenty-two, &c.*: three groups would be expressed by *three-tens*, which, abridged, is *thirty*, and so on. In this way, by ten names and their combinations, we are enabled to express any number of individuals less than ten groups of ten. Ten groups of ten must be expressed by *ten-tens*; but the reduplication of the same term in such a compound, if it can be called so, has rendered it more convenient to introduce a new name, and accordingly ten groups of ten are called a *hundred*.

(6.) Before we proceed further in explaining the language of numeration, we may observe that the individual objects which any number immediately expresses are called, with reference to that number, *units*.* Thus the counters in the example first given are the units composing each group; but in the collections of six groups subsequently formed the groups themselves are the *immediate units*. In like manner, in the decimal system of numeration, the *immediate units* of such a number as *seventy*, or *seven-tens*, are groups of ten, the units of each group being the individual objects, whatever they may be to which the number is applied. In the decimal scale, therefore, among the numbers

* UNUS, *one*

from one to one hundred there are two orders of units, the first being the individual objects, and the second the groups of ten.

When ten groups of ten are collected together, let us imagine them set apart and expressed by the word *hundred*; another collection of ten groups of ten may be placed beside them, and the number will be expressed by *two hundreds*; a third similar collection will make the number *three hundreds*, and so on, until ten such collections are made. It is evident that when that happens a collection will be formed in which *hundreds* play the same part as *tens* did with respect to *hundreds*, and as individual objects or *original units* did with respect to *tens*. A new name is imposed on such a number, and it is expressed by the word *thousand*.

(7.) These collections of ten-hundreds or thousands may be arranged in the same manner as hundreds, until we obtain ten such collections: analogy would then require the imposition of another name, but in our language the compound term *ten-thousand* has been adopted. When ten similar collections are made, analogy would again require a new name, but the language is again in discordance with the numerical theory, and the compound term *hundred-thousand* is used; ten groups of this class, which make up *ten hundred-thousands*, or a *thousand thousands*, is called a *million*.

Let us pause here and recapitulate the composition of this number. The objects or *units of the first order* are first conceived to be arranged in groups of *ten*, each group being considered a *unit of the second order*: these groups are then conceived to be arranged like the primary units in groups of ten, each group being called a *hundred*, these groups of a hundred being the *units of the third order*. It will be evident, that the number of units in the collection of each order ascending becomes rapidly less, the number of units of the second order being ten times less than the number of primary units, and the number of units of the third order being again ten times less than the number of units of the second

order. The groups of hundreds are now, in like manner, disposed in collections of ten, to form *units of the fourth order*, and these collections are called *thousands*; these again are collected in groups of ten, to form *units of the fifth order*, called *ten-thousands*; and these in groups of ten, to form *hundred-thousands*. Of these units there are but ten in the number contemplated: these ten, called a *million*, form a single *unit of the seventh order*.

(8.) A clear and distinct idea of the number expressed by the word "million" being thus formed, we may regard that number as a unit in the formation of higher numbers, and we may apply to two numbers, the units of which are millions, the same system of names which have been applied to the numbers rising from the primary units to a million. Accordingly, ten millions are as easily conceived and contemplated as ten units. The same may be said of a hundred or a thousand millions, and so on, until we have to express a million millions, when it is necessary for this number to introduce a new name; it is called a *billion*: in the same way, a billion being regarded as a new unit, the original names are applied until we reach a million *billions*, which is called a *trillion*. A trillion in the same manner becoming a unit is counted like the original units until we reach a million trillions, which is called a *quadrillion*, and so on. •

(9.) In this system of naming the first four orders of units are expressed, by distinct terms, *units*, *tens*, *hundreds*, *thousands*. The units of the fifth and sixth orders are expressed by compounding the names of those of the second and third orders with that of the fourth,—*ten-thousands*, *hundred-thousands*. The seventh order of units would be expressed by compounding the name of the fourth with itself,—*thousand-thousands*. To avoid this reiteration, the seventh order of units has received a distinct name,—*millions*. The names of the several orders of units after this are formed by compounding the previous names until we arrive at the thirteenth order of units, the compound name of which

would be “ millions of millions.” Here a new name is imposed,—*billion*. After this a new name is introduced at every sixth order of units.

It is impossible not to be struck with the admirable simplicity and efficiency of this system of names, and its beautiful adaptation to that mode of arrangement by which the mind is enabled to picture to itself numbers, however high, so distinctly, that any two differing only by a single unit are as clearly and unequivocally distinct as the most unequal numbers ; and by which the number of names is so limited, that every memory can easily retain them, and yet capable by composition of expressing a number of ideas which considerably exceeds the number of words commonly used in the most copious language.

(10.) The nomenclature of number in all languages clearly establishes the fact, that men have acquired their notions of number by the method of arranging and grouping individual objects, which we have already explained. The anomalies and irregularities which are sometimes found even in the nomenclature of the most civilised nations, and, indeed, often more in them than in the language of people less advanced, are indicative, not of any departure from this method, but of an inconstancy in the *radix of the scale*. It would appear that, in process of time, it had been found that a radix inconveniently great or inconveniently small had been used, and was by general consent abandoned, leaving, however, traces of its existence in some parts of the numerical nomenclature. In some cases, the radix appears to have been doubled at a former epoch, and we find traces in the language of one radix subordinate to another.

(11.) In the English language the nomenclature of number is purely decimal, subject, however, to a slight irregularity of structure, as already observed, from ten to twenty. The words “ eleven” and “ twelve” owe their origin to the combinations of the words *left one*, and *two left*, meaning that in counting a collection

of eleven or twelve individuals when the radix was counted off *one* was *left* in the one case, and *two* were *left* in the other: the formation of thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, is obvious: twenty is derived from the Gothic *twentig*, compounded of *twa*, two, and *tig*, ten; hence the formation of thirty, forty, &c. is apparent. The names used for units of the third and fourth orders, *hundred* and *thousand*, are from the German *hundert* and *tausend*: the names of the higher orders of units are from the Italian, *millione* signifying a great thousand, *i. e.* a *thousand thousand*, and hence *billione*, *trillione*, &c.

(12.) The French nomenclature is for the most part purely decimal. From ten to twenty the structure, like the English, is anomalous, but still formed from the ten preceding names. By strict analogy the numerals should be *dix-un*, *dix-deux*, *dix-trois*, *dix-quat*, &c. instead of *onze*, *douze*, *treize*, &c. The decimal system is observed from twenty (*vingt*) to sixty (*soixante*): here we find a vestige of an old vicenary scale. Seventy, instead of being *septante*, as the decimal system would require, is *soixante-dix* (sixty-ten); seventy-one, *soixante-onze* (sixty-cleven); seventy-two, *soixante-douze* (sixty-twelve), &c. Eighty, instead of being *octante*, is *quatre-vingt*, or four twenties, and ninety is *quatre-vingt-dix* (four twenties ten); ninety-one, *quatre-vingt-onze* (four twenties eleven), &c. Thus twenty becomes the radix of the system from sixty to a hundred.

(13.) The Greek and Latin nomenclatures are purely decimal, the term *εικοσι*, *twenty* in Greek, being, however, anomalous. *Κεντα* in Greek and *ginta* in Latin, when compounded, signify *ten*; and we have, accordingly, *viginti* (*biginti*) *twenty*, *τριάκοντα*, *triginta*, *thirty*, *τεσσαράκοντα*, *quadraginta*, *forty*, &c. &c.

(14.) The most perfect and symmetrical nomenclature for decimal numeration, so far as it is known to extend, is found in the language of THIBET. The first ten numbers are expressed in that language as usual by ten independent terms as follows:—

Cheic - <i>one.</i>	Tru - - <i>six.</i>
Gnea - <i>two.</i>	Toon - - <i>seven.</i>
Soom - <i>three.</i>	Ghe - - <i>eight.</i>
Zea - <i>four.</i>	Goo - - <i>nine.</i>
Gna - <i>five.</i>	Chutumbha <i>ten.</i>

In forming the numbers from ten to twenty the last two syllables of the name of ten are cut off, and that number expressed by the first syllable "chu:" to express eleven, twelve, &c. this syllable is succeeded by the names for one, two, three, &c., so that the numbers from ten to nineteen are expressed as follows:—

Chucheic (ten-one) <i>eleven.</i>	Chutru (ten-six) <i>sixteen.</i>
Chugnea (ten-two) <i>twelve.</i>	Chutoon (ten-seven) <i>seventeen.</i>
Chusoom (ten-three) <i>thirteen.</i>	Chughe (ten-eight) <i>eighteen.</i>
Chuzea (ten-four) <i>fourteen.</i>	Chugoo (ten-nine) <i>nineteen.</i>
Chugna (ten-five) <i>fifteen.</i>	

Twenty is expressed by *gnea chutumbha*, *two-ten*: the numbers from twenty to twenty-nine are expressed by the name for two, followed by the names for one, two, three, &c.

Gnea cheic (two-one) <i>twenty-one.</i>
Gnea gnea (two-two) <i>twenty-two.</i>
Gnea soom (two-three) <i>twenty-three.</i>
Gnea zea (two-four) <i>twenty-four.</i>
Gnea gna (two-five) <i>twenty-five.</i>
Gnea tru (two-six) <i>twenty-six.</i>
Gnea toon (two-seven) <i>twenty-seven.</i>
Gnea ghe (two-eight) <i>twenty-eight.</i>
Gnea goo (two-nine) <i>twenty-nine.</i>

It will be easily perceived that this system is, in fact, a transcript of the modern method of expressing numbers by the Arabic digits, and must certainly be admitted to be by far the most perfect nomenclature extant. There is, however, no evidence of the continuation of the nomenclature beyond the above names: these are given by Turner and other authorities.*

* Turner's Embassy to Thibet, 321.; Klaproth, Asia Polyglotta, 353.; Remusat, Recherches sur les Langues Tartares, 364., as quoted by Peacock in his article on Arithmetic.

(15.) The Sanskrit nomenclature is even more purely decimal than most of the modern European languages, inasmuch as a new name is introduced for every order of units. In the English and kindred languages, the grouping of number beyond a thousand is, strictly speaking, by thousands, so that the immediate radix of the system is a thousand, ten being subordinate to it. This, it is true, is a peculiarity to be ascribed rather to the names of high numbers than to our method of conceiving them, and does not at all belong to their expression by figures. In the Sanskrit, the first ten numbers are expressed by the following independent names, to which the Latin and its derivatives are evidently related : —

Eca	-	<i>one.</i>	Shata	-	<i>six.</i>
Dwau	-	<i>two.</i>	Sapta	-	<i>seven</i>
Traya	-	<i>three.</i>	Ashta	-	<i>eight.</i>
Chatur	-	<i>four.</i>	Nova	-	<i>nine.</i>
Ponga	-	<i>five.</i>	Dasá	-	<i>ten.</i>

The names for the successive orders of units are carried to a surprising extent : they are as follows :—

Eca	-	units.
Dasá	-	tens.
Sáta	-	hundreds.
Sahasra	-	thousands.
Ayuta	-	tens of thousands.
Lacsha	-	hundreds of thousands.
Prayuta	-	millions.
Cóti	-	tens of millions.
Arbuda	-	hundreds of millions.
Abja, or padma	-	thousands of millions.
C'harva	-	tens of thousands of millions.
Nic'harva	-	hundreds of thousands of millions.
Mahadpadma	-	billions.
Sáncu	-	tens of billions.
Jaludhi, or Samudra	-	hundreds of billions.
Antya	-	thousands of billions.
Madhya	-	tens of thousands of billions.
Parard'ha	-	hundreds of thousands of billions.

(16.) The Chinese, also, have a very perfect decimal system of numeration: the first ten numbers are, as before, expressed by ten distinct articulate sounds as follows: —

Yih	- - one.	Lyeù	- - six.
Irr	- - two.	Ts'hīh	- seven.
San	- - three.	Pāh	- - eight.
Sè	- - four.	Kyeú	- - nine.
Ngoo	- - five.	Shih	- - ten.

They have also distinct names for every order of units in the decimal scale, as far as hundreds of millions. which are as follows: —

Yih	units.	Eè	hundreds of thousands.
Shih	tens.	Chao	millions.
Pūh	hundreds.	King	tens of millions.
Ts'hyen	thousands.	Kyai	hundreds of millions.
Wan	tens of thousands.		

(17.) A decimal system of numeration prevails in all the Oriental tongues, in the languages of Greece and Rome, and in all the European tongues, including the Gothic. Mr. Peacock, in his admirable article on Arithmetic, in the *Encyclopædia Metropolitana*, quotes various authorities to show that the numeral nomenclatures of the native tribes of America are far more complete, both in structure and extent, than could be expected from the low state of civilisation of these people. Their systems of numeration are almost invariably the decimal, and seldom extend to less than the fourth order of units. The Knisteneaux, one of the principal hunting tribes of North America, who inhabit the northern shores of Lake Superior, have a decimal system of numeration which extends to 1000. The Sapibocones, a South American tribe, have decimal names to a like extent, but express a *hundred* and a *thousand* by *ten tens*, and *ten times ten tens*, without introducing any new numerical term after ten. The Algonquins, a kindred tribe of the Knisteneaux, speaking a dialect of the same language, possess several numerals in com-

mon, but have distinct names for hundreds and thousands. The numerals of the Hurons, at one time a numerous tribe of Upper Canada, inhabiting the shores of the lake of that name, are mentioned in a rare work by a Franciscan monk, G. Sagard, 1632, entitled "*La Grand Voyage des Hurons*," &c., dedicated to our Saviour. Although their language was so rude and artificial as to be destitute of adjectives, abstract nouns, or verbs of action, and incapable of expressing a negation without an absolute change of the word, yet it possessed a numeral nomenclature of regular structure, and formed on the decimal system. Equally complete systems of numeral language, all in the decimal scale, are found among the Indians of the Delaware, those who occupied the present district of New York, the former inhabitants of Virginia, and most of the central tribes of North America.*

(18.) The systems of numeration used by the tribes of South America are also generally constructed on the decimal radix, but they are frequently very limited in extent. They also form the names of the higher classes of units by such complex combinations, that the words expressing them appear almost impossible to be remembered. There is, however, one instance in which, in an original native South American language, viz. the ancient Peruvian, a decimal system of numeration exists, not less extensive than that of the Greeks and Romans, and one, indeed, which bears to ours a very curious analogy. There are distinct names for the second, third, and fourth orders of units,

Chunca	-	-	ten.
Pachac	-	-	hundred.
Huaranca	-	-	thousand.

But there occurs then no new term until we arrive at millions; ten thousand, is *chunca-huaranca*; a hundred thousand, *pachac-huaranca*; a million, *hunu*.

* Peacock on Arithmetic. Richardson, in Franklin's Journey. Mackenzie's Journey to the North Sea, Introduction. Humboldt, *Vues des Cordillères et des Monumens de l'Amerique*, 251. Momboddo, *Origin of Language*, 543.

This scale, like ours, proceeds after the first four orders of units, grouping the superior units by thousands, and not by tens ; or rather making the decimal system subordinate to the millesimal.*

(19.) People have a natural propensity to fix upon the different orders of units as landmarks, in the great ocean of number, from which to measure points in their vicinity. In the regular schemes which we have already noticed, numbers which terminate between two such points are always expressed by stating the number of units by which they *exceed* the units of the last order. Thus nineteen expresses nine above ten ; twenty-nine is nine above twenty, and so on. But it is obviously more natural for a people not sufficiently refined to appreciate the harmony of nomenclature, to express nineteen by *twenty wanting one*, and twenty-nine by *thirty wanting one*, and so on. We accordingly find this mode of expression in languages so numerous and remote as almost to preclude the possibility of the forms of expression being borrowed by one from the other. In the Sanskrit and Hindostanee nineteen and twenty-nine are expressed by *one less twenty*, and *one less thirty*, and similarly for higher numbers. In the Latin, *unus de viginti*, and *unus de triginta*, are more elegant than *novem decem*, and *viginti novem*. The same idiom prevails in the Greek. But what is more remarkable, we find a similar form of expression in various Oriental tongues. In the Malay language *nine* is expressed by *within one of ten* ; and *ninety-nine* by *within one of a hundred*. In the numeral language of the Sable-fur Ostiaks — a Siberian people inhabiting the banks of the Jenesei — nine is similarly expressed : — *eighteen* is expressed by *within two of twenty* ; *eighty*, by *within twenty of a hundred* ; *ninety-nine*, by *within one of a hundred*, and so on.†

Many other examples of this mode of expression, *by defect* from a complete number of the higher units, are found in the numeral terms of Iceland, Denmark, and several Oriental tongues.

* Peacock on Arithmetic. Humboldt, *Vues des Cordillères*, 252.

† Klaproth, *Asia Polyglotta*, 171.

(20.) We are not aware that there is an instance extant in any language, ancient or modern, of a complete numeral system formed upon a single radix other than the decimal. We have abundant examples, as already stated, of the occasional appearance of the quinary and vigesimal radices in scales which are chiefly decimal; and examples may be produced from the numerals of the Celtic dialects of a strange mixture of the quinary, denary, and vicenary systems. The Welsh, Erse, and Gaelic numerals, as far as ten, are expressed by independent words. In the Welsh, eleven appears to commence a period or phase of the decimal system, and is expressed by words signifying ten and one; the same formation continues to fifteen inclusive; but here the system assumes a quinary, or, perhaps, more properly, a quindecimal radix, sixteen being expressed by a word signifying fifteen and one; seventeen, fifteen and two, &c.: twenty is expressed by an independent term, and the scale here takes the vigesimal radix. The Erse and Gaelic scale, as far as twenty, is purely decimal; but at twenty, like the Welsh, the scale becomes vigesimal: from twenty to one hundred the scale of numeration is vigesimal in the three languages just mentioned; taking, however, the quindecimal radix at the numbers 35, 55, &c. in the Welsh. The Phenicians, among whom commerce, as is well known, was extensively cultivated at a very early period, had a scale of numerals constructed on the vigesimal radix. The first twenty numbers had distinct names, and the scale was continued upwards in the usual manner by compounding these names with twenty.* The intercourse which the Phenicians are known to have had with the southern parts of England, with Wales, and with Ireland, affords a satisfactory solution for the prevalence of the vigesimal radix in the numerals of these countries. The same solution may be given for the existence of the vigesimal scale in the Armorican and Basque dialects.

* Swinton, in *Philosophical Transactions*, 1758.

(21.) The vigesimal scale of numeration prevailed very generally among the Scandinavian nations, from which it is probable that we derive our method of counting by scores. This method, however, not being used at present to express abstract numbers, but only the application of number to particular classes of objects, can scarcely be considered as an existing instance of the vigesimal radix. The same may be said with respect to our method of counting by dozens, in which, however, we have three orders of units, — the primary unit, the dozen, and the gross, which is a dozen dozen. Were this ever applied to abstract numbers, it would furnish an example of the duodecimal radix.

(22.) Next to the decimal scale, the quinary is by far the most prevalent, as might be expected from its natural type, the fingers of one hand. The languages of some of the islands of the New Hebrides afford very perfect specimens of numeral systems constructed with this radix. The following are the numbers as far as ten of three of these : —

New Caledonia.		Tanna.		Mallicollo.	
Pārai	- one.	Rettec	-	- one.	Thkai - one.
Pā-rdo	- two.	Carroo	-	- two.	Ery - two.
Par-ghen	- three.	Kāhār	-	- three.	Erey - three
Par-bai	- four.	Kafā	-	- four.	Ebāts - four.
Pā-nim	- five.	Karirrom	-	- five.	Erihm - five.
Pānim-gha	six.	Ma-riddee	-	- six.	Tsukāi - six.
Pānim-roo	seven.	Ma-carroo	-	- seven.	Goory - seven.
Pānim-ghene	eight.	Ma-kāhār	-	- eight.	Goorey - eight.
Pānim-bai	nine.	Ma-kafā	-	- nine.	Goodbāts nine.
Parooneek	ten.	Karirrom-harirrom	-	- ten.	Sencām - ten.

There are, in each of these, five independent terms for the first five numbers. In the language of new Caledonia, the numbers from five to ten are expressed by *five-one, five-two, &c.* In the language of Tanna, they are expressed by *more one, more two, &c.*, and in the language of Mallicollo they are expressed by the words *one, two, &c.* combined with *tsu* or *goo*, the signification of the latter not being known.

(23.) In general the quinary system prevails among the most uncivilised tribes, and those who have little or no intercourse with other nations by commerce or otherwise. Among the least civilised of the Asiatic tribes abundant examples may be found of the use of this radix. Mr. Peacock quotes, on the authority of Klaproth, the languages of several tribes of Kamschatka, which, so far as they go, are purely quinary. The following are the numerals as far as ten in three of these : —

Onnen.	Ingsing.	Innen.
Hyttaka.	Gnitag.	Nirach
Ngroka.	Gnasog.	N'roch.
Ngraka.	Gnasag.	N'rach.
Myllanga.	Monlon.	Myllygen.
Onnan-myllanga	Ingsinagasit.	Innan-myllygen.
N'jettan-myllanga.	Gnitagasit.	Nirach-myllygen.
Ngrok-myllanga.	Gnasogasit.	Amorotkin.
Ngrak-myllanga.	Gnasagasit.	Chonatschinki.
Nyngytkan.	Damalagnos.	Myngyten.

(24.) When the quinary scale of numeration is carried to any considerable extent, it generally passes into the decimal or vigesimal systems. This, indeed, is a natural and necessary consequence of the inconvenience which would result from the introduction of so many new names as that system would require. The following example of the numerals of the Jaloffs, an African tribe, will illustrate this position : —

Ben, or benna.	Fook agh juorom.
Niar.	Fook agh juorom ben.
Nyet.	Nitt, or niar fook.
Nianet.	Fanever, or nyet fook.
Juorom.	Nianet fook.
Juorom ben.	Juorom fook
Juorom njar.	Temier.
Juorom nyet.	Niar temier.
Juorom nianet.	Djoone.
Fook.	Djoone agh temier.
Fook agh ben.	

The word here expressing five signifies the hand, plainly indicating the type of the system. It will be evident, on inspection, that the system is decimal, with the quinary radix subordinate to it.

(25.) The type of the vigesimal radix is indicated in some languages by the circumstance of twenty being expressed by the same word which signifies a *man*. The fingers and toes were evidently the original practical instruments of numeration; and after a number of objects had been counted, corresponding to the fingers and toes of one man, those of a second were referred to. Thus each *man* represented *twenty* of the objects counted. It will be easily understood from this, why the decimal and quinary scales are so frequently subordinate to each other and to the vigesimal; a hand, the two hands, and the hands and feet, furnished natural radices of the scale.

A complete examination of the numeral nomenclatures of various languages would afford results of the greatest interest, not only from the light which they would throw on the mutual relations and former intercourse of nations, but also on the general principles by which the notions of number are obtained. Such an investigation, however, would be unsuitable to the purpose, and inconsistent with the necessary limits, of the present work. Sufficient has, perhaps, been said to establish the fact that clear ideas of number can only be formed by grouping and arrangement, and that systems of nomenclature are always adapted to express such grouping and arrangement. It appears, also, sufficiently evident that the radix or base of such systems has had invariably a natural origin, and does not, as some have supposed, depend on any quality inherent in the abstract numbers which have been taken for such radices.*

(26.) There are many natural objects and circum-

* Those who desire to pursue this curious and interesting subject further should read the article on Arithmetic, by Mr. Peacock, already quoted, where they will be referred to numerous original authorities.

stances, as has been already stated, which would suggest *two* for the radix of the numerical scale. There is no instance, however, of any people adopting a binary system of numerals, and it is not difficult to perceive the reason of its general rejection. The number of independent terms with which the memory must be burdened in order to express the smallest extent of number necessary in the most common affairs of life, even among people not far advanced in civilisation, would be most inconvenient. The unit of the second order would be *two*, that of the third order *four*, of the fourth order *eight*, and so on. To express such a number as *sixty-three*, which in the decimal system is expressed by two words, would in the binary system require six, signifying severally thirty-two, sixteen, eight, four, two, and one. It is true that in higher numbers the names do not multiply so fast, but in these the inconvenience would be less important.

Like the duodecimal language, the binary is often applied to count particular objects, although never used in its abstract form. The words *pair*, *couple*, *leash*, *brace*, cannot be pronounced without calling to the mind of the hearers the various objects to which it is usual to apply such terms; these are even less abstract in their application than the terms *dozen* and *score*.

(27.) There are circumstances which would have rendered the number *twelve* a more convenient radix of numeration than ten; and there can be no doubt that if man had been a twelve-fingered animal, we should now possess a more perfect system of numeration than we do. Whatever be the radix of the scale, it would always be a convenience to be able to subdivide it with facility without resorting to the more refined expedient of fractional language; and in this respect *twelve* possesses much to recommend. Its half, third, fourth, and sixth parts can be all expressed by distinct numbers; of course the same applies to two thirds, three fourths, and five sixths of the radix. On the other

hand, *ten* allows only of its half, fifth, two fifths, three fifths, and four fifths being expressed by whole numbers. This advantage, however, would be greatly overbalanced by the inconvenience which would result from an attempt to change the generally established language of numeration.

CHAP. II.

ON THE METHOD OF EXPRESSING NUMBERS BY SYMBOLS OR FIGURES.

(28.) HAVING in the preceding chapter explained the manner in which clear and distinct ideas are formed of numbers, whatever be their magnitude, and the principles by which names are affixed to these ideas, and the *oral* nomenclature of number formed, we shall now proceed to consider the methods by which numbers are addressed to the eye, by means of signs, symbols, and written characters.

(29.) The most rude and inartificial method of expressing numbers by signs would evidently be by holding up as many fingers as there are units in the number to be expressed. The extent of such a scale of signs would in the first instance be limited to ten. Many and obvious contrivances would, however, soon suggest themselves for its extension, and we accordingly find INDIGITATION, or the art of expressing number by the fingers, practised to a considerable extent in different ages, and in various parts of the world.

Each finger having three joints, the fingers of one hand would suffice to count fifteen; and thus both hands used even in the most simple and inartificial manner would serve to count thirty. But if a quin-decimal scale were adopted, then, after the joints of the finger of one hand had been exhausted once, the superior unit fifteen might be expressed by the first joint on the other hand. Again, when the fifteen joints of the second hand had been twice counted, the two fifteens would be expressed by the second joint of the other

hand, and so on. In this way the joints of the fingers of both hands would enable us to count as far as fifteen times fifteen, or two hundred and twenty-five.

(30.) A system of digital reckoning was used by the ancients, by which they were enabled to count on the fingers as far as ten thousand. The first nine numbers, and ten, twenty, thirty, &c. to one hundred inclusive, were expressed by various inflexions of the fingers of the left hand. By such means the left hand alone was sufficient to count as far as one hundred: thus, to express seventy-five, the two inflexions expressing seventy and five should be exhibited. It is obvious that all that was necessary to be attended to in the formation of such a system of signs was, that each of the inflexions expressing ten, twenty, thirty, &c. should be possible to be made simultaneously with those which expressed the first nine numbers; and even if this were not accomplished, the signs might be made in succession. The fingers of the right hand expressed hundreds and thousands by the same inflexions as those by which the left hand expressed units and tens. It is obvious that the same system is capable of almost unlimited extension by changing the position of the hand or arm in making the signs. Thus, if the left hand expressed units and tens with the palm upwards, it might express ten thousands and hundred thousands with the palm downwards; and if the right hand expressed hundreds and thousands with the palm upwards, it might express millions and ten millions with the palm downwards, and so on.

(31.) The Chinese practise a method of reckoning on the fingers, in which one finger alone is made to express the first nine numbers, by placing the thumb nail on each joint of the little finger passing upwards from the palm of the hand to the extremity of the finger on the outside of the hand, then down the middle of the finger, returning to the palm, and, finally, upwards on the inside of the finger. The tens are

expressed in the same way on the next finger ; the hundreds on the succeeding one, and so on. In this way each finger expresses a distinct order of units, so that the four fingers and thumb include the first five orders of units ; and the hand is therefore capable of expressing all numbers under one hundred thousand.

The practice of indigitation prevails generally through the East, where commercial bargains are frequently made in that way instead of in writing, or by oral language.

(32.) To the expression of numbers by signs succeeded the more refined and artificial method of denoting and recording them by written characters. The symbols which would most naturally present themselves for this purpose were the letters of the alphabet: their form was familiar to every eye, and the order in which they stood being clearly impressed upon the memory of every one from childhood upwards, furnished an easy means of denoting the amount or value of particular numbers. The use of such characters originated in the East, where, indeed, the science of arithmetic may be considered as having had its birth, and where, at a very early epoch, it attained an astonishing degree of perfection, not only in its notation, but even in its complex operations. The Hebrews, Phenicians, and kindred nations expressed the first nine *digits*, as the numbers from one to nine are called, by the first nine letters of their alphabet ; as follows : —

א	Aleph	•	-	-	<i>one.</i>
ב	Beth	-	-	-	<i>two.</i>
ג	Gimel	-	-	-	<i>three.</i>
ד	Daleth	-	-	-	<i>four.</i>
ה	He	-	-	-	<i>five.</i>
ו	Vau	-	-	-	<i>six.</i>
ז	Zain	-	-	-	<i>seven.</i>
ח	Chet	-	-	-	<i>eight.</i>
ט	Teth	-	-	-	<i>nine.</i>

The next nine letters expressed *ten, twenty, thirty, &c.* to *ninety* inclusive; and the remainder of their alphabet, with some additional symbols, expressed *one hundred, two hundred, three hundred, &c.*

(33.) The Greeks adopted, with the utmost minuteness, this method of notation: so closely, indeed, that where a Hebrew letter was wanting in the Greek alphabet, its place was supplied by a symbol contrived for the purpose, and expressed by a name which signified that it held the place of the Hebrew letter. Thus for the Hebrew letter VAU there is no corresponding letter in the Greek alphabet, and the number *six*, which that letter expresses in the Hebrew, is accordingly expressed by the symbol ς, to which the name *επισημον βαυ* is affixed, meaning that it is the *sign of VAU*. In like manner, there are no Greek letters corresponding to KOPH and TSADI, and their places are accordingly supplied in the Greek scheme by the two symbols ς, and Ϸ, which are called *επισημον κοππα*, and *επισημον τανπι*, signifying the *sign for KOPH*, and the *sign for TSADI*. The Greek numerals are as follows:—

1 α	10 ι	100 ρ
2 β	20 κ	200 σ
3 γ	30 λ	300 τ
4 δ	40 μ	400 υ
5 ε	50 ν	500 φ
6 ς	60 ξ	600 χ
7 ζ	70 ο	700 ψ
8 η	80 π	800 ω
9 θ	90 ς	900 Ϸ

It was customary to distinguish the letters when employed as numerals from the same when employed in the ordinary way by placing an accent on them. The same letters were made to express thousands by placing ι below them:

1000 α	4000 δ	7000 ζ
2000 β	5000 ϵ	8000 η
3000 γ	6000 ς	9000 θ

(34.) The Greeks also employed another method of expressing number inferior to the above. This method consisted in denoting the different classes of units by peculiar letters. Thus α denoted the primary or original units, Π denoted fives, Δ tens, H hundreds, X thousands, and M ten thousands. The last four letters being the initial letters of the Greek words signifying *tens*, *hundreds*, *thousands*, and *ten thousands*. The numeral letter placed before any one of these expressed the number: thus, ζX , was *seven thousand*, &c.

(35.) The ancient numerals of Arabia, Syria, Persia, and other nations were formed in the same manner from their alphabets. The European languages, including the Russian, Gothic, Scandinavian, and Sclavonic tongues also expressed their numerals by their respective alphabets.

(36.) The Roman numeral notation is formed by combinations of the following symbols: —

I.	-	one.
V.	-	• five.
X.	-	ten.
L.	-	fifty.
C.	-	one hundred.
D.	- °	five hundred.
M.	-	one thousand.

In some cases five hundred is expressed by IC , and a thousand by CIC . The following table will explain the way in which these symbols are combined to express numbers: —

Units.		Tens.		Hundreds.		Thousands.		Tens of Thousands.	
I	1	x	10	c	100	CID or M	1000	CCID or \bar{x}	10,000
II	2	xx	20	cc	200	MM or \bar{ii}	2000	CCIDCCID	} 20,000
III	3	xxx	30	ccc	300	MMM or \bar{iii}	3000	or \bar{xx}	
III	} 4	xxxx	} 40	cccc	} 400	MMMM	} 4000	&c. &c.	
or		or		or		or \bar{iv}			
IV	} 5	XL	} 50	CD	} 500	ID or \bar{v}	} 5000		
V		L		DC					
VI	6	LX	60	or	} 600	IDM or \bar{vi}	} 6000		
				DC					
				DCC	} 700	IDMM	} 7000		
VII	7	LXX	70	or		or \bar{vii}			
				IDCC	} 800		} 8000		
				DCCC		IDMMM			
VIII	8	LXXX	80	or	} 800	or \bar{viii}	} 8000		
				IDCCC					
				DCCCC	} 900		} 9000		
VIII	} 9	LXXXX	} 90	or		IDMMMM			
or		or		IDCCCC		or \bar{ix}			
IX	} 9	XC	} 90	or	} 900		} 9000		
				CM					

It will be perceived that when a symbol of lesser value is placed before one of greater value, it has the effect of subtracting its own value from that which follows it. Thus IV. signifies *five minus one*, and VI. signifies *five plus one*; IX. signifies *ten minus one*; and XI. *ten plus one*, &c. When a line is drawn above any numeral, it makes its units signify thousands. Thus while IX. signifies *nine primary units*, \bar{IX} . signifies *nine thousands*. It is also a curious circumstance, and worthy of notice, that while the numeral language of Rome is purely decimal, its numeral symbols appear to have the quinary radix subordinate to the decimal. Thus a new symbol is introduced at five, which reappears at fifteen, twenty-five, &c., another new symbol is introduced at fifty, and another at five hundred.

(37.) If these various systems of numeral symbols be carefully attended to, it will be perceived that the chief source of their complexity and inconvenience is

the necessity of making the symbol express not only its actual numeral amount, but also the order of units. In fact, a new set of symbols becomes necessary for each order of units. The contrivances in the Greek numerals for signifying thousands by the same set of numerals with the accent placed below them forms a slight approach to what would have been a more symmetrical and effective system of notation. It cannot fail to excite surprise that, having seized on the decimal system so perfectly in the mind, the mental classification which was adopted did not suggest a system of symbols more analogous than those which were so long in general use. The notions of ten, twenty, thirty, forty, &c. would have naturally suggested a set of symbols for their expression similar to those used for the primary units from one to nine, modified by an accent or some other similar means. Another inflexion or mark would have enabled the same nine characters to express the hundreds. It may probably be said that our astonishment at the imperfection of the old numerical notation arises from our extreme familiarity with the very perfect system which has since been adopted. But although something may be allowed for this, yet there is in the very process of mind, by which only we are enabled to acquire clear ideas of number, something which would have prompted analogous systems of names for the units of succeeding orders. The observations contained in the preceding chapter cannot, we think, be denied to afford sufficient proof of this.

(38.) A people among whom the arts and sciences had been so highly cultivated, and where the development of the human mind was so advanced as among the Greeks, having once obtained a clear mental view of the decimal system of numeration, could not, one would have supposed, have failed to have formed some system of notation, bearing at least as close an analogy to the idea intended to be expressed as the following. Supposing the first nine digits to be expressed as already explained by the nine characters.—

α'	β'	γ'	δ'	ϵ'	ζ'	η'	θ'	
1	2	3	4	5	6	7	8	9

then the same numbers, when their units are of the second order, or when they signify tens, might have received a double accent; thus,—

α''	β''	γ''	δ''	ϵ''	ζ''	η''	θ''	
10	20	30	40	50	60	70	80	90

Again, when they signified units of the third order, or hundreds, they might have received a triple accent; thus,—

α'''	β'''	γ'''	δ'''	ϵ'''	ζ'''	η'''	θ'''	
100	200	300	400	500	600	700	800	900

and so on. Although this would have been immeasurably inferior to the system of notation now universally adopted, still it would have given enormously increased power to their arithmetic, by suggesting more easy and expeditious methods of calculation.

(39.) In his investigations respecting the Egyptian hieroglyphics, Dr. Young has explained the numeral system used in them. It appears to be exclusively decimal, no subordinate radix appearing in the scale. The primary unit is expressed by \square , and the first nine digits are expressed by simple repetitions of the primary unit. The unit of the second order, or ten, is expressed by \cap , and twenty, thirty, &c. by repetitions of this. In the same manner, a hundred and a thousand are expressed by \bigcirc and Σ , and hundreds and thousands, as far as nine, expressed by simple repetitions of this. Thus, 432 would be expressed as follows:—

$\bigcirc\bigcirc\bigcirc\bigcirc\cap\cap\cap\square\square$

(40.) But the triumph of the art of calculation, and that to which mainly the modern system of numeral computation owes its perfection, consists in the “device of place;” by which all necessity for distinguishing the nature of the units signified by any symbol is super-

seded. Like many other inventions of the highest utility, this, when known, appears to arise so naturally and necessarily out of the exigencies of the case, that it must excite unqualified astonishment how it could have remained so long undiscovered.

Let us imagine a person possessing a clear notion of the decimal method of classifying number, being desirous to count a numerous collection of objects by the help of common counters. He will, probably, at first pursue the method already described as practised by the savage tribes of Madagascar.* The objects to be counted being passed before him one by one, he places a counter in a box A for each object that passes; but presently the counters in A become so numerous, and form so confused a heap, that he finds it as difficult to form an idea of their number as he would of the objects themselves which he wishes to count. Being able, however, to form a distinct and clear notion of ten counters, he pauses when he has placed the tenth counter in the box A, and withdraws all the counters from it, placing a single counter in the box B, to denote that ten objects have passed. He then recommences his tale; and, as the objects continue to pass before him, places counters in the box A, and continues to do so until ten more objects have passed, and ten counters are again collected in A: he withdraws this second collection of ten counters from A, and places a second counter in B; signifying thereby that two sets of ten objects have passed. Recommencing a third time, he proceeds in the same way, and, when ten have passed, withdraws the counters from A, and places a third counter in B: he continues in this manner, placing a counter in B for every ten which he withdraws from A. If the objects to be counted be numerous, he finds, after some time, that the counters would collect in B so as to form a number of which he would still find it impossible to obtain a clear notion. For the same reason, therefore, that he allows no more than ten counters to accumulate in A, he adopts the same expedient with

* See page 5.

respect to the box B. When ten sets of objects have been counted, he finds that ten counters have collected in B: he withdraws them, and places a single counter in the box C, that counter being the representative of the ten withdrawn from B, each of which is itself the representative of ten withdrawn from A. The single counter in C will thus express the number of objects in ten sets of ten; and such a number as already explained is called a hundred.* When one hundred objects have passed, there will therefore be only a single counter expressing it placed in the box C. The objects to be counted continuing to pass, the computer proceeds as before, placing counters in the box A, withdrawing them by tens, and signifying the collections withdrawn by placing single counters in B, until ten counters again collect in B; these are withdrawn, and a second counter placed in C. Let us now conceive the three boxes inscribed with the names of the units signified by the counters which they respectively contain. It will be obvious that, by the aid of twenty-seven counters, all numbers under a thousand may be expressed. Thus, nine hundred and ninety-nine would be expressed by placing nine counters in each box: the nine counters in the box C would stand for nine hundreds; those in the box B for nine tens, and those in A for nine original units.

It will be sufficiently evident that the same method may be continued to any extent. A fourth box, D, inscribed thousands, may be provided, in which a single counter will be placed for every ten counters withdrawn from C; and a fifth, E, inscribed ten thousands, in which a single counter will be placed for every ten withdrawn from D, and so on. Under such circumstances, more than nine counters could never collect in any box.

We have here supposed the counters to be all similar to each other, and not bearing on them any character or

* To demand a *proof* that ten times ten make a hundred, betrays an ignorance of the true meaning of numerical terms. The *definition* of the word hundred is ten tens.

mark ; but, as we have inscribed the several boxes with the names of the order of units which the counters they contain express, there is no reason why the counters themselves may not be inscribed with a character by which a single counter may be made to express any number of units from one to nine. Let us, then, suppose the computer furnished with an assortment of counters, incised with the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 : when he would express the number of units in each box, instead of placing in it several individual counters, the number of which might not be easily perceived, he places in the box a single counter, inscribed with a character which expresses the number of single counters which would otherwise be placed in the box. Thus, instead of leaving six individual counters in a box, he would place in it a single counter, marked with the character 6 : by such an arrangement, the number to be expressed would be always evident on inspection, as here exhibited :—

F	E	D	C	B	
4	3	5	7	3	
Hundreds of Thousands	Tens of Thousands	Thousands.	Hundreds	Tens	Units

Four hundred and thirty-five thousand seven hundred and thirty-one.

Having adopted such a *method of reckoning, he would naturally, for convenience, always arrange the several boxes in the same manner ; and very speedily the PLACE in which the box stood would indicate to him *the order of the units* which it contains : thus he would be at no loss to remember that the second and third boxes from the right would always contain tens and hundreds, and the like of the others. The formal inscription, *units, tens, &c.* would, therefore, become unnecessary ; and since, by the method of incising the counters with figures, no more than one counter need be placed in any box, the boxes themselves would be dispensed with, and it would be sufficient to place the

counters one beside the other, the PLACE of each counter indicating the rank of units which it signifies.

(41.) A slight difficulty would, however, occasionally present itself. Suppose that it should so happen that, when the last object to be counted passed, the tenth counter was placed in the box C, according to the system explained: all the counters would be withdrawn from C, and a single counter placed in D, or a counter containing a figure higher by one than that which was placed in it before. When the complete number is expressed, the box C would, in this case, contain no counter. When the boxes are superseded, and the counters alone used, the place of the third counter from the right would be unoccupied, and the number would be expressed by the counters thus:—



The space between the counters inscribed 5 and 3 here shows the absence of the counter which would express hundreds; but in placing the counters, through negligence or otherwise, it might happen that the two counters which should thus be separated by a space, might be brought so close together, that, in reading the number, the space might be overlooked; in which case, the counter inscribed 5 would erroneously be supposed to express 5 hundreds. To provide against such an error, let us suppose blank counters to be supplied, and one of these placed in the position which would be occupied by an empty box: the above number would then appear thus:—



and no mistake could possibly ensue.

The next step in the improvement of this method would be to abandon counters altogether, and immediately to write down the figures which would be in-

scribed on them if they were used ; these figures being written in the same order in which the counters were supposed to be placed. In this case, a character would become necessary to signify the place of a blank counter, wherever such a one might occur : the character which would be naturally adopted for this purpose would be 0 ; and the above number would then be 435031.

Such is the system of numerical notation which has obtained in every part of the world an acceptance, the universality of which can only be attributed to its admirable simplicity and efficiency.

(42.) If we examine this notation in comparison with other systems, we shall find it distinguished by two peculiarities ; first, the expression of the first nine numbers by single characters ; and secondly, by making the *same characters* express units of all orders, by adopting a certain *invariable arrangement*, and introducing a tenth character (0), *to maintain this arrangement undisturbed* when units of any particular order or orders happen to be wanting in the number to be expressed.

That the honour of the invention of a system which produced such important effects, as well on the investigations of science as in the common concerns of commerce, should be claimed by many contending nations, is what would naturally be expected. We accordingly find various opinions advocated, as well respecting the people with whom this system originated, as with respect to those who first had the honour of introducing it into Europe. It is, however, agreed on all hands, that the method of expressing number by nine figures and zero, with the method of giving value to these by local position, so as to enable the same characters to express the successive orders of units, was brought into Europe immediately from Arabia ; and hence the common figures now used are distinguished from the Roman numerals by the name *Arabic figures*.

(43.) All Arabian authors on arithmetic appear to agree that the first writer of that country upon this system of arithmetic was Mohammed ben Muza, the

Khuwarezmitte, who flourished about the year 900. This writer is celebrated for having introduced among his countrymen many important parts of the science of the Hindoos, to the cultivation of which he was devotedly attached; and, among other branches of knowledge thence derived, there is satisfactory evidence that this species of arithmetic was one. From the time of Mohammed ben Muza, the figures and modes of calculation introduced by him were generally adopted by scientific writers of Arabia, although a much longer period elapsed before they got into common popular use, even in that country. They were always distinguished by the name *Hindasi*; meaning, the Indian mode of computation. Alkindi, the celebrated Arabian writer, who flourished soon after Ben Muza, wrote a work on arithmetic, entitled "*Hisabu l' Hindi*;" meaning, the arithmetic of the Hindoos. In addition to this evidence of its Indian origin, we have the unvarying testimony of all subsequent Arabian writers. But there is internal evidence from the system itself, as compared with the mode of writing and reading Arabic, which furnishes a still more decisive proof of its origin. The mode of writing practised in Arabia was like ours, from left to right; whereas in writing those figures they proceed in the contrary direction, in the manner practised by the Hindoos.

(44.) There is extant positive evidence of the existence of this arithmetic in India at least four centuries before the time of Ben Muza. We possess, in our own language, translations of two treatises on arithmetic, mensuration, and algebra, which are highly prized in Hindostan: they are entitled the "*Lilâvati*" and "*Vijaganita*," and are the works of Bhascara. Mr. Colebroke has fixed the age of Bhascara, on satisfactory evidence, about the middle of the twelfth century. Bhascara, in his work, quotes the authority of a former writer, fragments of whose treatise on arithmetic are still extant, named Brahme-gupta. Mr. Colebroke has also shown that this author flourished in the beginning of the seventh century. Brahme-gupta has again quoted

a still more ancient authority, Arya-bhatta, as the oldest of the uninspired writers of that country. It appears that this writer flourished at a period not later than the beginning of the fifth century. Now, it is remarkable, that none of these Hindoo authors claim, either for themselves or their predecessors, the invention of the method of numeration by nine digits and zero, with a method of giving value by position, but always mention it as being received from the Deity; from which we may infer, that it was practised in that country beyond the limits even of tradition.

(45.) At the beginning of the eleventh century, the use of the Arabic notation had become universal in all the scientific works of Arabian writers, and more especially in their astronomical tables. The knowledge of it was, of course, communicated to all those people with whom the Moors held that intercourse which would lead to a community of scientific research. In the beginning of the eleventh century, the Moors were in possession of the southern part of Spain, where the sciences were then actively cultivated: in this way the use of the new arithmetic was received into Europe first in scientific treatises. A translation of Ptolemy was published in Spain in 1136, in which this notation was used; and after this period it continued in general use for the purposes of science.

(46.) Notwithstanding the knowledge and practice of this superior notation by scientific men, the Roman numerals continued to be used for purposes of business and commerce for nearly three centuries; and it was only by slow and gradual steps that the improved notation prevailed over its clumsy and incommodious predecessor. The first attempt to introduce it for the purposes of commerce was made by a Tuscan merchant, Leonardo Pisano; who, in 1202, published a treatise on arithmetic with a view to introduce it among his countrymen. Leonardo had travelled into Egypt, Barbary, and Syria: his father appears to have held some office in the custom-house at Bugia in Barbary, where he

represented the interests of the merchants of Pisa. The son there learned the method of Hindoo arithmetic; and, struck with its superiority over that to which he had been accustomed, he determined that his countrymen should no longer be deprived of the benefit of it. He accordingly published his treatise in the Latin language; in which he professes to deliver a complete doctrine of the numbers of the Indians: — “*Plenam numerorum doctrinam edidi Hindorum, quem modum in ipsa scientia præstantiorem elegi.*”

(47.) The date of this work has been disputed, and it has been contended that it is the production of a later age. This supposition is, however, attended with some difficulties. It is evident, from the work itself, that at the time it was written Algorithm (the name given to the Indian arithmetic) was not known or practised in Italy; yet, it is certain that treatises on arithmetic with Arabic numerals were common in Italy, and well known during the whole of the fourteenth century. Those who dispute the date of Leonardo's work, refer it, nevertheless, to a period so late as the fifteenth century. But there is another argument still more decisive against such an hypothesis. In the fifteenth century, to which the treatise of Leonardo is referred, the Italian language had long superseded the Latin. In all ordinary works, indeed, the general use of the Italian language instead of the Latin, commenced about the middle of the thirteenth century. It is therefore impossible to suppose that Leonardo would write, for the avowed purpose of benefiting his countrymen engaged in commerce, a treatise on arithmetic in a language of which they would probably understand as little as an Englishman of the present day would have understood the language used before the time of Chaucer.

(48.) The work of Leonardo is referred to by Lucas de Burgo, in 1484, and by all subsequent writers, as being the first means of introducing the Arabic notation into Italy. A considerable period, however, was neces-

sary to introduce this system into the common business of life. The extensive commerce maintained by the Italian states directed their attention to the subject at an earlier period than other nations; and although, for scientific purposes, the date of the introduction of the Arabic numeration into Spain is earlier than that of its appearance in Italy, yet its use for the common business of life prevailed at a much earlier period among the Italian states than in any other nation of Europe. To the exigencies of Italian trade, we owe the formal subdivision of arithmetic under the various heads of the Rule of Three, Profit and Loss, Exchange, Interest, Discount, &c. &c.

(49.) Although the artifice of expressing the successive orders of units by the same signs, arranged in different positions, was undoubtedly the invention of the Hindoos, and to the combination of this principle with the adoption of the nine digits and zero to express the digital numbers, is due the great perfection and efficiency of the present system, yet the old method of computation, practised among the Romans, and subsequently used in Europe until the introduction of the Arabic notation, contained traces of this principle. The application of it was, unquestionably, very inferior, because it was used merely for computation, and not for recording or expressing numbers.

(50.) Among the Greeks the elements of arithmetical knowledge were taught upon a board called an *ABAX*. On this board progressive rows of counters were placed, which consisted of pebbles, pieces of ivory or coins. The Greek word for pebble is *psephos*, and hence the word *PSEPHIZEIN*, to *compute or reckon*: the Latin word for pebble is *CALCULUS*, and hence *calculari* to reckon, and our term to *calculate*.

(51.) From the *abax* of the Greeks the Romans derived their *ABACUS*: this was in like manner a board on which pebbles (*calculi*) were placed, and which by various arrangements were made the instruments of calculation. "The use of the *abacus*," says Professor

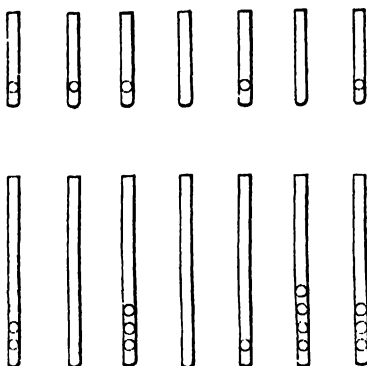
Leslie," formed an essential part of the education of every noble youth. A small box or coffer called a *loculus*, having compartments for holding the calculi or counters, was considered a necessary appendage. Instead of carrying a slate and satchel, as in modern times, the Roman boy was accustomed to trudge to school loaded with those ruder implements, his arithmetical board, and his box of counters."

(52.) The form of the abacus was subsequently improved: instead of the perpendicular lines or bars, of which it first consisted, the board had its surface divided by sets of parallel grooves, by extended wires, or by successive rows of holes. It was easy to move small counters in the grooves, to slide perforated beads along the wires, or to stick large knobs or round-headed nails in the different holes.* A representation of such an instrument is here given, *fig. 1*. There are seven long parallel grooves in the lower row, over which are severally written the names of the units, which the counters they contain express. To prevent the necessity of using a great number of counters another set of shorter grooves are placed above the former, in which a single counter is equivalent to five in the groove below. Thus, four counters in the groove below, and one in the groove above, count nine; three below and one above count eight, and so on. The first groove proceeding from the right to the left, expresses the primary units, the second tens, the third hundreds, and so on; so that the last of seven grooves expresses millions. It will be evident, therefore, that any number expressed by the common Arabic figures will be expressed by such an instrument in exactly the same manner, only inserting in the successive grooves the number of counters corresponding to the digits which occupy the places severally. Thus, if we would express the number 7,580648, we should do it in the following manner:—In the lower groove of units we should place three counters, and in the upper groove one; in the lower groove of tens, four,

* Leslie's *Philosophy of Arithmetic*, p. 95.

and none in the upper ; in the lower groove of hundreds, one, and one in the upper ; in the grooves of thousands, none ; in the lower groove of ten thousands, three, and one in the upper ; in the lower groove of hundred thousands, none, and one in the upper ; in the lower groove of millions, two, and one in the upper.

Fig. 1.

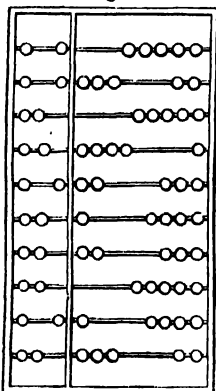


(53.) It will be evident, that this method of expressing numbers is subject to no other limit than the number of grooves which may be provided in the abacus. That number should always be the same as the number of digits by which the number could be expressed in the Arabic notation. Thus, to express ten millions would require eight figures in the Arabic notation, and eight grooves in the abacus. In fact, the inscription of I, X, C, &c., over the grooves successively, is no more necessary in the abacus than the inscription of units, tens, hundreds, &c., over the successive figures of a common number. By general consent, the first groove on the right being used for units, the second for tens, &c. the inscription of the value of the grooves would be unnecessary, and their *position* would become the indication of that value.

We have here, then, the principle of *value by position* distinctly practised; and if the Romans had thought of expressing by nine characters in writing what in calculation they here expressed by counters placed in the grooves, they would undoubtedly have hit upon the Arabic method of notation and computation.

(54.) A curious coincidence is observable between the Roman abacus and a calculating instrument called the *SWAN-PAN*, used by the Chinese. A representation of this instrument of calculation is given in *fig. 2*. It

Fig. 2.



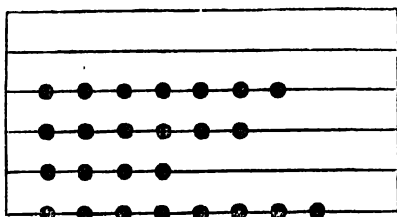
consists of ten parallel wires unequally divided: there are five beads on each of the longer, and two on each of the shorter divisions. In representing numbers on this instrument, it is held so that the wires are horizontal; and the values of the beads increase in decuple progression downwards. The beads on the top wire express the primary units; on the second the tens, and so on. This instrument is admirably adapted for Chinese calculation, since the subdivision of their mea-

sures, weights, and money is made on the decimal system: hence the calculator may select at pleasure any bar for the primary units; in which case, the bars above it will express the subdivisions by tenth parts. "Those arithmetical machines," says professor Leslie, "have been adopted in China by all ranks, from the man of letters to the humblest shopkeeper, and are constantly used in all the bazaars and booths of Canton and other cities, being handled, it is said, by the native traders with a rapidity and address which quite astonish the European factors."*

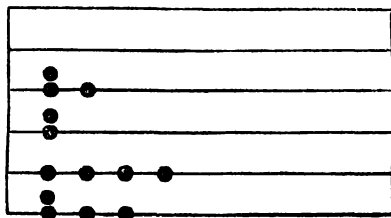
(55.) From the abacus of the Romans was derived

* Leslie's Phil. of Arith. p. 98.

the methods of reckoning by counters practised throughout the whole of Europe during the middle ages, and the use of which was continued until it was superseded by the improved arithmetic now in use. The calculating board or frame was abandoned, and in its place a number of parallel lines, drawn on a board or other surface, were substituted. The lowest of these lines was the line of primary units; the second, proceeding upwards, expressed tens; the third hundreds, and so on. A plan of this kind being placed before the calculator, he expressed the number by placing counters on the parallel lines. Thus, to express 7648, he would arrange his counters as follows:—

Fig. 3.

The difficulty of catching at a glance the number of counters placed on the several lines, when they are numerous, led to the adoption of an expedient, probably suggested by the use of the Roman numeral V: five counters on any line was expressed by a single counter

Fig. 4.

placed immediately above it. The number, therefore, otherwise expressed in *fig. 3.*, would, according to this arrangement, be expressed as in *fig. 4.*

On the lower line, the counter immediately above counts *five*; which, added to the *three* on the line, makes *eight* for the units; the tens being less than *five*, remain as before; the *six* hundreds are expressed by *one* counter on the third line and *one* above it; and the seven thousands are expressed by *two* counters on the fourth line and *one* above it.

(56.) The methods of performing the various arithmetical operations by these means were extremely simple and obvious, and addressed themselves to the understanding even more plainly and forcibly than do the rules of the arithmetic practised at the present day. We shall hereafter refer to them more fully.

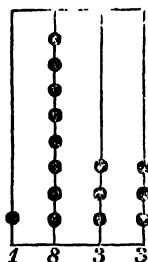
During the middle ages, it was usual for merchants, accountants, and judges, who arranged matters of revenue, to appear on a covered *banc*, so called from an old Saxon word signifying *a seat*. Before them was placed a flat surface, covered by a black cloth, divided by parallel white lines into perpendicular columns, and these again divided transversely by lines crossing the former, so as to separate each column into squares. This table was called an *EXCHEQUER*, from its resemblance to a chess board; and the calculations were made by counters placed on its several divisions in the manner and according to the principles which we have just explained.

(57.) A money-changer's office or shop was commonly indicated by a sign of this chequered board suspended. This sign afterwards came to indicate an inn or house of public entertainment, probably from the circumstance of the innkeeper also following the trade of a money-changer; a coincidence which is still very common in sea-port towns.

(58.) The transition from the method of expressing numbers and making computations upon them by counters to the improved method now in use, seems to

us easy and natural, although the change took more than three centuries in being effected. Let us suppose the rows of counters expressing the different orders of units, instead of being placed horizontally, as we have just described, to be placed in an upright or vertical position, the units' column standing in the first place on the right, the tens' succeeding it towards the left, and so on. Such a scheme would differ in no respect from the arrangement of the abacus of the Romans, and

Fig. 5.



would differ only in position from the computing board of more modern times. The number 1833 would be expressed as in *fig. 5*.

The difference between the two methods would thus be reduced to the device of expressing all collections of units less than nine by single characters, and of marking a blank line by 0.

(59.) From what has been explained in the preceding chapter respecting the nomenclature of number, no difficulty will be found in appropriating the names of the successive orders of units to the successive places of figures, beginning from the right. When the figures are numerous, however, some inconvenience and difficulty may be found in perceiving at a glance the order of units expressed by the first figure on the left. If it were customary to announce in spoken language the amount of numbers, by beginning with the primary units, and then ascending through tens, hundreds, &c., a number expressed in figures could be read without any difficulty, however numerous its figures might be. But as the first figure to be announced is the highest order of units in the number, before we can express it, we must perceive the total number of figures, and ascertain the order of units which the first figure on the left expresses. For the purpose of facilitating this, it has been usual, when high numbers are expressed, to point or distribute them into periods of three or six, some-

times marked by leaving a wider space between every third or sixth figure, and sometimes by introducing a comma.

(60.) The distribution into periods of three places corresponds to the nomenclature used for high numbers in France, where a *thousand million* is called a *billion*, a *thousand billion* a *trillion*, and so on; but according to our nomenclature, a *billion*, as already explained, signifies a *million millions*, a *trillion* a *million billions*, and so on. It is, therefore, more consistent with our nomenclature to distribute the figures expressing numbers into periods of six. The advantage of this will be seen by attempting to read the following number without such distribution:—

3 5 7 6 2 0 0 1 3 7 9 6 1 0 2 4 6 8 9.

In its present state this number cannot be read without counting off the figures from the units' place, calling them units, tens, &c., until we arrive at the first figure on the left; but if they be distributed into periods of six as below, we perceive at once that the first figure standing in

3 576200 137961 024689

the fourth period, here consisting of one digit, signifies trillions, and the number would be read thus;—3 trillions, 576200 billions, 137961 millions, 024689.

(61.) This system of numeral notation being well understood, it will be perceived that every digit, of which a number is composed; has two distinct values, which it will be convenient to denominate the *absolute* and the *local value*. The absolute value of a digit is that value which it has when it occupies the place of the primary units, and is, therefore, the number of individuals which it expresses; the *local value* is, as the term intimates, that value which it derives from its position or place in the number to which it belongs. Thus, in the number 365, the absolute value of 3 is three units, the local value is 3 hundreds.

(62.) The object of 0 being merely to fill a place, so as to mark the position of other digits, and thereby to give them their local value, this character is distinguished from the other digits which possess absolute value. The digits which possess absolute value, viz. 1, 2, 3, 4, 5, 6, 7, 8, 9, are called *significant digits*, to contradistinguish them from 0.

(63.) The character 0 is called nought (*nothing*), and is also called *cipher*, a term which is derived from the Arabic word *tsaphara*, which signifies a *blank* or *void*. The uses of this character in numeration are so important that its name *cipher* has been extended to the whole art of arithmetic, which has been called *to cipher*, meaning, *to work with figures*.

(64.) If a 0 be added on the right of any number, its effect will be to remove each digit of that number one place to the left. Now, since the local value of the digits increases from right to left in a decuple proportion, this effect will be equivalent to increasing the value of every digit tenfold, and therefore multiplying the number by 10. Suppose the number in question is 999; if we add 0 to this, and make it 9990, the 9 which before occupied the units' place is transferred to the tens, and signifies 9 tens instead of 9 units; the 9 which filled the place of tens is transferred to the place of hundreds, and signifies 9 hundreds instead of 9 tens; and the 9 which filled the place of hundreds is passed to the place of thousands. Thus the local value of each of the three digits is increased 10 times, and since the 0 itself has no value, the latter number is exactly 10 times the former.

(65.) For like reasons the addition of two ciphers to the right will multiply the number by 100, since it transferred the digit which occupied the units' place to the hundreds' place, that which occupied the tens' place to the thousands', and so on.

(66.) In general, the addition of three ciphers will multiply a number by 1000, the addition of four by 10,000, and so on. On the other hand, ciphers pre-

fixed to a number, or placed to the left of it, produce no effect on its value, because they do not change the place of any of its digits; the digit which before occupied the place of units still retains the same position and the same may be said of the others. Thus, 999, 0999, 00999 have all the same value. We must infer therefore, that the first figure on the left of a number can never be 0, since in that position the hought has no signification.

(67.) A significant digit placed to the right of a number has the same effect in multiplying it by 10 that a cipher would, and for the same reason; but, besides this, the value of the significant digit is added to the number; thus, if we place 9 to the right of any number, that number will be multiplied by 10, and 9 will be added to the product. Let 3 be the number; 9 annexed to the right of this makes the 3 signify 30, and adds 9. In the same manner 9 annexed to the right of 30 makes the 3 signify 3 hundreds instead of 3 tens, and also adds 9, making the number 3 hundreds and 9 (309). For the same reason two or more significant digits added to the right of a number have the same effect in multiplying it by 100, 1000, &c. as the same number of ciphers would have; but they also add their own value to the number so multiplied. Thus, if we annex 99 to the right of 1, we shall change the 1 into 100, just as we should do by annexing two ciphers; but to this number will be also added 99.

(68.) A significant digit added to the left of a number makes no change in the value of any other digit in the number, but it adds to the number the local value which that digit acquires from the position in which it is placed. Thus, if to 99 we prefix 1, the 99 will still retain the same value as before, but the 1 prefixed will, in the position given to it, signify a hundred, and thus 100 will be added to the former number.

Having thus developed, with some detail, the principles on which the nomenclature and notation of arith-

metic depend, we shall proceed in the following chapters to explain the various processes by which numbers are combined or separated. The most simple of these operations are ADDITION and SUBTRACTION, out of which will naturally arise two other operations, MULTIPLICATION and DIVISION.

CHAP. III.

ADDITION.

(69.) WHEN two or more numbers are added together, the number which is obtained by such addition is called their SUM. Thus, if 3, 5, and 2, added together, give 10, then 10 is said to be the *sum* of 3, 5, and 2.

If the method of performing arithmetical operations practised before the adoption of the present arithmetic were incommodious in practice, limited in its powers, and inadequate to the wants of a highly advanced state of society, it was not nevertheless destitute of other advantages, which, even at the present day, should recommend it as an instrument at least of illustration. The operations of arithmetic performed by it did not require the results of previous calculations in the form of tables to be committed to memory, as is necessary in all the elementary operations of arithmetic now practised. But what is of greater importance to our present purpose is, that the *rationale* of the process was so palpable that it could not fail to be perceived by any one capable of managing the counters. In fact, *the rules required no proofs*, the reasons of the process being self-evident. So far is this from being the case with our present arithmetic, that many hold (in which, however, we do not concur) that the reasons of the rules of arithmetic are incapable of being made intelligible to children at the early period of life, at which circumstances render it necessary that they should learn the practice of them. Although, however, we feel assured that a skilful teacher, or even an ordinary teacher, when assisted by a well

written work of instruction, would be capable of making the reasons of the arithmetical operations understood by a child of common capacity, at the age at which these operations are usually taught, yet it is certain that such a task would be incomparably more difficult than to teach the same child the reason of the operations when performed with counters. Indeed, the best method of making the reasons of the rules of computation apparent seems to be this : — to teach the rules first by the old method of counters placed upon lines, and then to represent the same process in figures. This would at least be necessary in the first rules, viz. addition and subtraction. When these are well understood, the minds of most children would, perhaps, be enabled to comprehend the inference of the process of multiplication and division from them without the further use of illustration by counters. Nevertheless, we cannot too strongly recommend such a method to teachers ; it renders the ideas of numbers and their mutual relations clear and definite, and will disabuse the learner of the pernicious habit, so commonly contracted, of acquiring a flippancy in terms where the mind has no corresponding ideas whatever, or no distinct ones.

(70.) Let us suppose that it is required to add together the following numbers, 1768, 2804, 9999, and 5407. As there are four orders of units in these numbers, we shall mark upon paper four vertical columns, included by parallel lines marked A, B, C, and D, intended to receive counters expressive of the number of units of each order, in the number to be added. In the column of primary units, marked A, we first put eight counters for the eight units in the first number ; in the column marked B, we put six counters for the tens in that number ; in the column marked C, we put seven counters for the hundreds ; and in the column marked D, we put a single counter for the thousands. We then cut off each number by a line, and in the same way express the second and succeeding numbers.

From the column A we now withdraw ten counters, and put a single counter into the column B; this evi-

	D	C	B	A	
	○	○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○ ○	1768
	○ ○	○ ○ ○ ○ ○ ○ ○ ○		○ ○ ○ ○	2804
	○ ○ ○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○ ○ ○	9999
	○ ○ ○ ○ ○ ○ ∅	○ ○ ○ ○ ∅	∅ ∅	○ ○ ○ ○ ○ ○ ○	5407
X	E				Y
	∅	○ ○ ○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○ ○	19978

dently makes no change in the total number expressed by the counters, since one counter in the column B is equivalent to ten in the column A. We next withdraw ten more counters from A, and put another counter into B; there will then remain eight counters in A, two counters being added to the column B: the eight counters remaining in A are moved down and placed in the same column under the line X Y; the two counters added to the column B are marked thus ∅.

We now withdraw ten counters from B, and place a single counter in C, which is distinguished from the others by the same mark ∅: seven counters remain in B, which are in like manner moved down and placed in

the column B, under the line X Y. Two sets of ten counters are now withdrawn from the column C, and two additional counters placed in the column D, marked as before: the remaining counters in C will be nine, which are moved down and placed as before under the line X Y, in the same column: 10 counters are now withdrawn from D, and a single counter placed under the line X Y, in an additional column E: the 9 counters which remain in D are brought down as before, and placed under the line X Y.

It will be perceived that for every 10 counters which are withdrawn from any column in this process, a single counter is placed in the next column to the left, a circumstance which makes no change in the amount of the total number expressed by the counters, since 10 counters in any column are equivalent to a single counter in the column next on its left. The counter thus added to the column on the left is said to be *carried*. Thus for 20 counters withdrawn from A, we carried 2 to the column B; and for 10 withdrawn from B, we carried 1 to the column C; and for 20 counters withdrawn from C, we carried 2 to the column D; and, finally, for 10 withdrawn from the column D, we carried 1 to the additional column E. The counters which remain in each column, after as many tens as possible are withdrawn, being always less than 10 in number, are brought down, and occupy the same column in the sum as they did in the numbers themselves. In the present case, therefore, the sum is 19978.

(71.) It will be evident, upon the slightest consideration, that if a calculator be furnished with a sufficient number of counters, he can by this method add together any collection of numbers, however great, without the possibility of error, and without previously committing to memory any table of addition. Let us now see how the same object would be attained, if the several numbers to be added were expressed by figures instead of counters. For the sake of clearness, we shall preserve

the same scheme of parallel lines, placing the figures in the squares previously occupied by the counters.

D	C	B	A
1	7	6	8
2	8	0	4
9	9	9	9

E

1 | 9 | 9 8 | Sum

It will be necessary, in the first place, to withdraw the tens from the column A, but to do this is not so easy a matter as it was when counters were used. We must now, in the first place, ascertain the total amount of the four figures in the column A; and this can only be done by having previously calculated, by the aid of counters, or some other such means, the sum which would be obtained by the addition of every two single figures. In fact, a TABLE OF ADDITION has become necessary, which must either be referred to for the purpose of ascertaining the sum of every two digits, or must be committed to memory, so that such sum may be recollected when the knowledge of it is needed. Not to interrupt our process of investigation, we shall assume, for the present, that such a table has been committed to memory: the computist then knows that 7 added to 9 gives 16: he withdraws the 10, and bears in recollection that 1 is to be carried to the column B; the other 6 he adds to 4, which gives another 10, so that 2 must be carried to B; the remaining 8 he writes in the units' place of the sum immediately under 7. He now adds to 9 in the column B the 2 which he has carried; this gives 11; he sets apart 10, bearing in mind that 1 is to be carried to C, and he adds the remaining 1 to 6, placing 7 in the tens' place of the sum. The 1 which he carries to C he adds to 4, which gives him 5; this added to 9

gives 14 ; he sets apart the 10, and adds 4 to 8, which gives 12 ; he sets apart the second 10, and adds 2 to 7, which gives 9 ; this 9 he places in the hundreds' place of the sum. Having reserved 2 tens in the column C, he carries 2 to the column D ; these 2 added to 5 give 7, which added to 9 gives 16 ; reserving 10, he adds 6 to 2, which gives 8, and this he adds to 1, which gives 9 ; he writes 9 in the thousands' place of the sum. Having reserved one ten from the column D, he writes 1 in the ten thousands' place of the sum, and the calculation is complete.

(72.) It is clear that in this case there are two sources of possible error, which do not exist in the method of counters. One is the possibility of not perfectly remembering the true sum of every two digits. Considering how small the number of pairs of digits, the sums of which it is necessary to remember, is, this, perhaps, can scarcely be regarded after a little practice as a source of probable error : the other is the probability of forgetting the number of tens reserved in each column, and therefore of carrying a wrong number to the succeeding one. When the numbers to be added are considerable, this is a source of very frequent error, even with practised arithmeticians, and different computers adopt different means of registering the number to be carried. If the column of figures to be added be not great, the fingers of the left hand will serve as a register for each 10 which is reserved, provided the addition of the column does not amount to 60 ; and by using the fingers a second time, this method may enable him to register them as far as 100. Other computers find it more convenient, instead of reserving and registering the tens, to add the column directly up, performing each addition mentally : thus in the units' column of the number just given, the computer would say 7 and 9 are 16, and 4 are 20, and 8 are 28. By practice this becomes no very difficult matter ; but still it is liable to error upon a momentary relaxation of attention in the computer.

(73.) Whatever method the computer may adopt for the purpose of registering the carriages from column to column, he must at all events commit to memory the sums of every possible pair of single digits which can be required to be combined by addition.

(74.) The sign $+$ is used to express the operation of addition, and when it occurs between two numbers it is intended to express their sum: thus $5 + 2$ means that number which is found by adding 2 to 5.

(75.) The sign $=$ means *equal*, or *is equal to*, or *makes*; thus $5 + 2 = 7$ means that 2 added to 5 makes 7, or that the sum of 2 and 5 is 7. The reader will find it convenient to render himself familiar with the use of these abbreviations.

(76.) In the opposite table of addition the single digits are combined by pairs in every possible manner; so that, if it be committed to memory, the sums of every pair of digits will be known, and the computer will be in a condition to solve any questions whatever in mere addition.

(77.) In the first column of this table are expressed the sums of the different pairs which follow the numbers in the cross rows:—thus 6 is obtained by the addition of 1 and 5, 2 and 4, or 3 and 3: 9 by the addition of 1 and 8, 2 and 7, 3 and 6, or 4 and 5; and the same with the others. Although it is probable that the results here tabulated will be already familiar to the mind of every reader of the present treatise, yet we consider it right, in this and other cases, distinctly to explain the principle or rationale by which such results, however familiar, are verified and proved, and to distinguish clearly between what depends on the conventional nomenclature of number, and that which is matter of inference; or, in other words, to mark out clearly the boundary where the province of definition terminates, and the operation of demonstration begins. That 1 and 1 make 2 is a matter of definition; it is, in fact, the meaning of the word or figure 2. In the same manner that 1 and 2 make 3, is the same as saying

TABLE OF ADDITION.

2	1 + 1				
3	1 + 2				
4	1 + 3	2 + 2			
5	1 + 4	2 + 3			
6	1 + 5	2 + 4	3 + 3		
7	1 + 6	2 + 5	3 + 4		
8	1 + 7	2 + 6	3 + 5	4 + 4	
9	1 + 8	2 + 7	3 + 6	4 + 5	
10	1 + 9	2 + 8	3 + 7	4 + 6	5 + 5
11	2 + 9	3 + 8	4 + 7	5 + 6	
12	3 + 9	4 + 8	5 + 7	6 + 6	
13	4 + 9	5 + 8	6 + 7		
14	5 + 9	6 + 8	7 + 7		
15	6 + 9	7 + 8			
16	7 + 9	8 + 8			
17	8 + 9				
18	9 + 9				

that 3 is the number next above 2, and, therefore, this is a definition. In the same way it will appear that the second column of the above table as far as $1+9$ inclusive tells nothing that was not already made known in the explication of the nomenclature and notation of number. The next term of the column, however, is a matter of inference: the sum of 2 and 9, whatever it be, exceeds the sum of 1 and 9 by a single unit; but the sum of 1 and 9 is 10, by definition: therefore, the sum of 2 and 9, being 1 greater than 10, must be 11. In the same manner, the sum of 3 and 9, being 1 greater than the sum of 2 and 9, must be 12. For a like reason the sum of 4 and 9 must be 13, and so on. Thus the remainder of the second column is derived by an easy and obvious inference from the preceding part of it.

(78.) The third column may be derived from the second, by very simple reasoning. It will be observed, that the two numbers added in the third column are one of them greater, and the other less, than the two which are added in the second: thus, opposite 6 we have $1+5$; succeeding $1+5$ in the same row we have $2+4$. Now, if we suppose the numbers to express counters, $2+4$ may be derived from $1+5$ by transferring a single counter from the 5 to the 1; this transfer can make no change in the total amount of the counters, and, therefore, whatever be the sum of 1 and 5, the same must be the sum of 2 and 4. But we have already seen that the sum of 1 and 5 is 6, and, therefore, 6 must also be the sum of 2 and 4. The same reasoning will show that the sum of 1 and 6 is the same as the sum of 2 and 5; that $1+7 = 2+6$, and so on. Thus the third column is directly inferred from the second.

(79.) By a like process of reasoning the fourth column may be inferred from the third, for there, in like manner, of the two numbers added, the first exceeds and the other falls short of the preceding numbers by one. Opposite 12, for example, we find $4+8$ and $5+7$: if we consider them as representing heaps of counters, the

latter is evidently derived from the former by transferring a single counter from the heap of 8 to the heap of 4.

The fifth column is derived from the fourth, and the sixth from the fifth, by exactly the same reasoning ; and thus the results of this table, however familiar, are reduced to the rigorous test of demonstration, and receive the same validity and certainty as the conclusions of geometry.

(80.) In teaching the first principles of arithmetic to children, who must needs at a very early age commit to memory the computations of the above table, how much more effectual would the instruction be rendered if a teacher would occasionally take the trouble (pleasure we would rather say) of addressing himself to the understanding of his little pupil, as well as to the memory ! What could be more easy than to provide a parcel of counters, and dispose them in heaps or in rows, and make the child verify with them all the above results ? the very playthings of the child might thus be made, as they ought always to be, instruments of instruction.

(81.) A computer, as has been already observed, finds it frequently convenient to be able to add with facility and despatch a single digit to a number consisting of two or more digits ; when the above table of addition has been committed to memory, such an addition is almost as easy as the addition of only two digits, and, indeed, in some cases is quite as easy. Thus, suppose we wish to add 4 to 35, we have only to add 4 to the units in 35, which gives 39 ; a process which is as easy and expeditious as adding 4 to 5. If, however, the addition of the single digit to the units produce a sum greater than 9, it will then be necessary to carry 1 to the tens ; but as such an addition never can produce a sum greater than 18, it never can be necessary in this case to carry more than 1 to the tens. Thus, for example, if we wish to add 9 to 35, we should get 30 and 14 ; the 4 would take the units' place, and

1 would be carried to the 3, so that the sum would be 44. In such additions, therefore, whenever the sum of the units exceeds 9, the last figure only is to be retained for the units, and 1 is to be added to the tens. The young student must always be made to practise additions of this kind, so as to perform them with facility and despatch, before he is introduced to more complex questions.

(82.) As in all arithmetical operations as at present performed, there is a certain liability to error,—a liability which is great in proportion to the complexity of the question and the number of numerical quantities which it involves,—it is in the last degree desirable, if not absolutely necessary, that the computer should possess some means of verifying his work, and that those who employ the results of the computation should be able to discover whether they be erroneous. When computers are used for any practical purpose, as in the calculations made for almanacks, tables of insurance, logarithms, trigonometry, &c., probably one of the most effectual and best methods of verification is to make different computers work independently of one another, and to compare their results: if they obtain the same result by the same set of operations, it may be assumed to be all but mathematically certain that the conclusion is correct; for to suppose it erroneous, it would be necessary to assume that the different computers committed the *same number of errors*, that these errors were *exactly the same in magnitude*, and fell upon the *same figures*: such a supposition would outrage all the rules of probability.

(83.) The same practical principle may be applied in the instruction of youth, whether in numbers or individually. If a number of pupils be instructed together, let the same question be proposed to them all, or to any convenient number of them, and take care that those who are working the same question shall not have communication with each other. If their answers agree, it may be assumed that all are correct; if not,

the discretion of the master and his knowledge of the pupil will probably direct him in the selection of those on whom it may be advisable to impose the labour of revision.

(84.) Even if the instruction is conveyed to an individual pupil, it will not be difficult to propound the same question at different times, under forms so different that it will not be known to be the same: the results may then be compared, and if they are identical, the solution may be assumed to be correct. Thus if it be proposed to add together several numbers, the figures which occupy the places of any order of units may be transposed at pleasure, so as in appearance, and indeed in reality, to vary the numbers added together, but at the same time to make no change in their sum. We have here stated several varieties which may be given to the question in addition which has been already propounded.

1904	2808	1467
9899	1407	5988
5467	9969	2829
2708	5794	9694
<hr/>	<hr/>	<hr/>
19978	19978	19978
<hr/>	<hr/>	<hr/>

In one the column of units consists, reading downwards, of 4978: in the original number it consists of 8497, the same figures differently arranged, but of course making up by addition the same number of primary units. In the tens' column the only significant digits are 9 and 6, and the same may be said of the tens' column of the original number. If the column of hundreds and thousands be likewise examined, they will be found in like manner to consist of the same figures as in the original number, varied only in their order.

(85.) This mode of testing questions in addition proposed simultaneously to several pupils, or at different times to the same pupil, may be further varied by in-

creasing any figure in any column, and at the same time decreasing another figure in the same column by the same amount: thus in the units' column of the first example, instead of 8 and 4 we may write 7 and 5, and instead of 9 and 7 we may write 8 and 8, which will convert the units' column into 7588, reading downwards, instead of 8497; and the same principle will apply to the columns of units of superior orders. In this way the same question may be disguised under such a variety of forms that it is impossible the pupils can recognise its identity.

(86.) But this method of verification by the master may be carried still further: besides varying the arrangement of the figures in each column, he may also add 1 or 2 to some one figure in every column, so that the answers furnished by one set of pupils could be found by adding 1 or 2 to every digit in the answer furnished by another set.

(87.) These methods afford the teacher easy and expeditious means of ascertaining whether the work of his pupils be correct, without the trouble and loss of time attendant on performing the same calculation himself. As occasional errors are, however, incidental on all arithmetical calculations made with figures, it is desirable that the computer himself shall possess some means of verifying his work. Several methods of effecting this present themselves, adapted to the different classes of questions to which the work is applied: these may generally be resolved into performing the work twice by different methods, and trying the coincidence of the results; if they be both correct they must be both identical; if not identical, one or other must be erroneous, and the work requires revision.

(88.) Let us take the first example already given, for the addition of numbers: it appears that by adding the several columns in the ordinary way the sum was found to be 19978; now let the addition be performed, beginning in each column at the top instead of at the bottom, and try whether the same sum will be obtained;

or cut off the top line, and add the three lines below it; then add the top line to the sum thus obtained: the result should be still 19978. The numbers to be added may also be divided into two, the first two lines being first added, and then the last two lines; the two sums thus obtained added together should produce the same final sum. We have here subjoined several numbers to be added together, and their addition in the usual way gives us as a sum 4519; we then add the first three, and find their sum to be 1750; the next three added gives 837; the next three gives 542, and the last two give 1390: these four numbers added together ought to give the same sum as the number first produced, and they accordingly do so. Had any error been committed in making the original addition it is in the last degree improbable that the same error exactly should be made in performing the several partial additions, or in adding the partial sums together.

$$\begin{array}{r}
 357 \\
 431 \\
 962 \\
 \hline
 479 \qquad 1750 \\
 301 \\
 57 \\
 \hline
 9 \qquad 837 \\
 200 \\
 333 \cdot \\
 \hline
 489 \qquad 542 \\
 901 \\
 \hline
 \qquad 1390 \\
 \hline
 4519 \qquad 4519 \\
 \hline
 \end{array}$$

(89.) It is the invariable practice in modern arithmetic to add upwards, and from right to left. This is not, however, necessarily connected with the principle of the operation, nor was it always the mode of practising it. In the Hindoo arithmetic, as given in the *Lilāvati*, the method pursued is different in detail,

though identical in principle. It may not be uninteresting to recur occasionally to these old and obsolete methods, since it will assist us in perceiving what it is in the modern processes which is arbitrary, and what essential. We shall thereby the more clearly perceive and remember the principle which is involved in every operation. Let us suppose that the numbers to be added are 4, 12, 37, 8, 64, and 201; the process would be as follows:—

$$\begin{array}{rcl}
 \text{Sum of the hundreds, } 2 & - & 2 \\
 \text{Sum of the tens, } 1, 3, 6 & - & 10 \\
 \text{Sum of the units, } 4, 2, 7, 8, 4, 1 & - & 26 \\
 \hline
 & & 326 \\
 \hline
 \end{array}$$

Or, if we proceed from the inferior to the superior orders of units, it would be as follows:—

$$\begin{array}{rcl}
 \text{Sum of the units, } 4, 2, 7, 8, 4, 1 & - & 26^{\wedge} \\
 \text{Sum of the tens, } 1, 3, 6 & - & 10 \\
 \text{Sum of the hundreds, } 2 & - & 2 \\
 \hline
 & & 326 \\
 \hline
 \end{array}$$

(90.) It is evidently unimportant in what order the process of addition is conducted, if only the numbers carried be duly attended to. Let us suppose that the following numbers are required to be added together:—

$$\begin{array}{r}
 24605 \\
 68979 \\
 30895^{\wedge} \\
 47638 \\
 32756 \\
 87104 \\
 \hline
 68747 \\
 22328 \\
 \hline
 291977
 \end{array}$$

We shall commence our proceedings at any proposed column, suppose the column of hundreds: adding up this column we find that it makes 37; we place 7 under the column of hundreds, and three under the column of thousands, leaving above it a space to receive the number obtained by the addition of that column: let us next add the column of ten thousands; we find that its sum is 26, and we accordingly place 6 under it, and 2 in the place of hundred thousands, leaving a space above: in the same manner add the units' column; we find that the sum is 37, and we accordingly place 7 under the units' column and 3 under the tens', with a space above; the tens being now added give 24, and we place a 4 under the column of tens, and a 2 under the column of hundreds: the column of thousands are now added, which give us 28, and we place accordingly 8 under the column of thousands, and two under the ten thousands. The lower line now exhibits the several numbers carried from the units to the tens, from the tens to the hundreds, &c.; these numbers being added to the numbers above give the total sum.

(91.) It is obvious that in the ordinary way of performing the process of addition, the lower line is added *mentally* to the upper: thus when we add the units' column we carry 3, which being added to the sum of the tens' column, gives the 7 for the tens' place, and leaves 2 to be carried, and so on; the actual process being an abridgment by mental calculation of that which is here written down.

(92.) If it should so happen that the sum of all the digits forming each column of units, those of the highest order excepted, should be less than 9, it is a matter of absolute indifference whether the process of addition begins from the column of the highest order of units, and proceeds to the lowest, or begins at the lowest and proceeds to the highest; or, in fact, in what order it may be taken. In the following example we may obtain the same sum with the same facility and

expedition, whether we proceed from left to right or from right to left; or begin at the middle column and proceed first to the left and then to the right, or perform the operation in any other order.

$$\begin{array}{r}
 93041 \\
 72130 \\
 62323 \\
 \hline
 227494 \\
 \hline
 \end{array}$$

Although it is probable that what has been already said will make the principle of addition sufficiently intelligible, yet it may not be useless in this and similar cases to lay down, in formal terms, the several steps of the process, which require to be attended to.

General Rule for Addition.

(93.) In order to add several numbers together, write each of these numbers one under the other in such a manner, that the units of the same order shall stand in the same vertical column; that is, that the units of one number shall be immediately under the units of another, the tens under the tens, the hundreds under the hundreds, and so on: then add together the digits found in the units' column; if their sum be expressed by a single digit, write this figure under the units' column, and commence the same process with the tens' column. But if the sum of the digits in the units' column be greater than 9, it must in that case be expressed by more than one figure; write the last figure only under the units' column, and carry to the column of tens as many units as are expressed by the remaining figures; this number must be added to the column of tens. Proceed in the same manner with the column of tens, and so with every column, proceeding from right

to left. When the column of the highest order, which is always the first on the left, has been added, including the number carried from the preceding column, if the sum be expressed by a single figure, place that figure under the column, but, if it be expressed by more figures than one, write these figures in their proper order, the last under the column, and the other preceding it.

CHAP. IV.

SUBTRACTION.

(94.) WHEN two unequal numbers are proposed, there is a certain number, which, being added to the lesser would make it equal to the greater; and it is evident that if the amount of the lesser be taken from the greater, the same number would remain. The arithmetical process by which such a number is discovered, is called **SUBTRACTION**. This operation, therefore, may be considered under two points of view, either as the means of diminishing the greater, by the amount of the less, or of increasing the less, until it becomes equal to the greater. It was evidently under the former point of view that the operation received the name of **SUBTRACTION**; and the same way of considering the process gave the name **MINUEND**, or "number to be diminished," to the greater of the two proposed numbers; and **SUBTRAHEND**, or "number to be taken away," to the less. In order, however, to obtain a full and clear perception of the principle of the operation, it will be necessary that we should consider it under both points of view.

(95.) In commercial arithmetic, the result of the operation is commonly called the **REMAINDER**, a term which also implies that the operation is one by which the greater number is diminished by the amount of the less: in the mathematical sciences, the result of the operation is called the **DIFFERENCE** of the two numbers, thereby implying merely that it is that by which the greater exceeds the less.

(96.) Let us suppose the minuend **A** and subtrahend **B** to be expressed by counters, there being as many counters in the row **A**, as there are units in the minuend, and as many in the row **B** as there are units in the subtrahend.

A . . . 000000000000
 B 00000000

If we remove the counters by pairs, beginning from the right, until all the counters of the subtrahend are taken away, and also the counters immediately above them in the minuend, the counters which remain will be evidently the difference between the original numbers, and will be the remainder which would be obtained in the process of subtraction. In the present case, this remainder is 4. Now, suppose that, instead of removing the counters by pairs, we add counters to the subtrahend B, until we fill all the vacant places below the counters in A; the two numbers A and B will then be equalised by increasing B, and the number of counters necessary to effect this will be 4.

(97.) Before we pass to the consideration of more complex cases, we shall take occasion to observe that by adding the same number of counters to A and B, their *difference* will not be changed, and of course their *remainder*, obtained by the process of subtraction, will continue the same. This will be evident if we observe the effect of adding a counter to A and B on the right. At present eight of the counters in the row A, have eight immediately below them in the row B: if we add another counter to the right in the row A, and another also to the right in the row B, the number in each row will be increased by one; but still the counters to the left in A, which have none below them in B, continue the same, and these are the counters which form the remainder in the process of subtraction. We, therefore, infer, that, without affecting the result of the operation, we may always *add* the same number to the subtrahend and minuend, and for like reasons we may also *deduct* the same number from both.

(98.) Let us now suppose that it is required to subtract 2345 from 4689: we shall consider the units of each order expressed by counters in the subjoined scheme: —

D	C	B	A	
0 0 0 0	0 0 0 0 0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	4689
0 0	0 0 0	0 0 0 0	0 0 0 0 0	2345
0 0	0 0 0	0 0 0 0	0 0 0 0	2344

If we consider that the object of the operation is to take from the superior number the amount of the inferior, and to place the remainder in the third line, we shall proceed as follows: — Remove 5 counters from the 9 in the column A of the minuend, and transfer the remaining counters of the column to the remainder: we have thus taken from the units of the minuend as many as are contained in the units' place of the subtrahend. In the same manner we shall take 4 from the tens of the minuend, and transfer the remaining 4 to the remainder. Of the 8 tens in the minuend, we have, therefore, removed 4: in the same manner, from the 6 hundreds of the minuend, we take 3 hundreds, which is the number in the subtrahend, and place the remaining 3 counters in the hundreds of the remainder. In the minuend there are 4 thousands, and in the subtrahend 2; taking 2 from the former, we place the remaining 2 in the remainder: the remainder, therefore, is 2344.

(99.) If we consider the operation under another point of view, we should obtain the same remainder in the following manner: — Place counters in the units' place of the remainder, until the number placed there, added to the number in the units of the subtrahend, shall make up the units of the minuend: thus, there are 5 counters in the column A of the subtrahend; if we put 4 in the same column of the remainder, these

4, added to the 5, will make up 9, which is the number of counters in the column A of the minuend. In the same manner, if we place 4 in the column B of the remainder, these 4, added to the 4 in the same column of the subtrahend, will make 8, which is the number contained in the same column of the minuend. There are 3 counters in the column C of the subtrahend; and if 3 more be placed in the same column of the remainder, the two added together will make 6, which is the number of counters in the same column of the minuend. In the same manner, 2 counters placed in the column D of the remainder, added to 2 in the same column of the subtrahend, give 4, the number of counters in the same column of the minuend. Thus, the units of each order in the remainder, added to the units of the same order in the subtrahend, produce a sum or total equal to the number of the units of the same order in the minuend.

(100.) The example just given is attended with circumstances which afford a facility in the operation of subtraction, which are not found generally to exist: A difficulty frequently presents itself, which will be perceived in the following example.

(101.) Let it be required to subtract 2987 from 4345.

D	C	B.	A	
0 0 0 0	0 0 0	0 0 0 0	0 0 0 0 0	4345
0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0	2987
0	0 0 0	0 0 0 0 0	0 0 0 0 0 0 0 0	1358

If we consider the operation as one by which the lesser number must be taken from the greater, we must proceed as follows:—Since the number of counters in the column A of the minuend is less than the number in the same column of the subtrahend, it is evident that the latter cannot be taken from the former: to remove this inconvenience, let us take away one of the 4 counters in B, and place 10 more counters in A. It is evident that this will make no real change in the value of the number expressed by the counters in the first row, since 1 counter in B is equivalent to 10 in A: we may thus have 15 counters in A, and 7 under it in the subtrahend. Taking 7 from the 15, there will remain 8, and 8 counters are accordingly put in the column A of the remainder: 3 counters now remain in B, under which are 8, in the subtrahend; since it is impossible to take 8 from 3, we must have recourse to a contrivance similar to that just resorted to. A single counter is taken from C, and 10 counters placed in B: there are now 13 counters in B, and 8 under them in the subtrahend: taking away 8 of the 13 in B, there will remain 5; and 5 counters are accordingly placed in the column B of the remainder. There are now 2 counters in C, under which are 9 in the subtrahend: as no subtraction can be performed in this case, the same expedient is adopted as before; a single counter is withdrawn from D, and 10 counters more placed in C: there are now 12 counters in C, from which 9 must be taken; 3 will remain, and 3 counters are accordingly placed in the column C of the remainder. There are now 3 counters in the column D, having 2 counters under them in the subtrahend: 2 being taken from the 3, one will remain; and 1 is accordingly placed in the column D of the remainder: the remainder is, therefore, 1358.

(102.) Now, it will be observed, that the difficulty which here presents itself, arises from this circumstance. although the subtrahend, *as a whole*, is a less number than the minuend, and therefore may be subtracted from

it, yet the units of inferior orders in the former happen to be greater than those of the same orders in the latter ; consequently it is impossible to subtract the units of each order in the subtrahend from those of the same order in the minuend. In the first example above given this difficulty did not present itself, because the number of units of each order in the minuend was greater than the number of units of the same order in the subtrahend. The expedient by which this difficulty is overcome, consists in borrowing, as it were, a unit of a superior order, and adding its value to the units of the inferior order, which are deficient in number.

(103.) If we consider the operation in another point of view, the question will be, to discover the number, which, being added to the subtrahend, would produce a sum equal to the minuend. Since no number of counters added to 7 would produce 5, we suppose such a number added to it as would produce 15 ; that number being 8, we place 8 counters in the column A of the remainder ; but since the addition of these two produce 15, it will be necessary to carry 1 to the tens' place : we must, therefore, add 1 counter in the column B of the subtrahend ; there will thus be 9 counters, to which a number must be added, which would give 4 for the sum : as this is impossible, we add a number which would give 14 ; that number being 5, we place 5 counters in the column B of the remainder, and carry 1 to the column C. There will now be 10 counters in the column C of the subtrahend, to which such a number must be added as would cause 3 counters to be placed in the sum ; that number being 3 itself, 3 counters are placed in the remainder, and 1 is carried to the column D : there will then be 3 counters in the column D of the subtrahend, which would require the addition of 1 to make up the number in the minuend ; 1 is, therefore, placed in the remainder, and the result is 1358.

(104.) The student will now have no difficulty in applying the above reasoning to any numbers expressed

in figures. Let it be required to subtract 2431 from 4679. The units of each order are here regarded separately, and subtracted one from the other: thus taking 1 unit from 9, 8 remain; taking 3 tens from 7 tens, 4 tens remain. In the same manner, taking 4 hundreds and 2 thousands from 6 hundreds and 4 thousands, 2 hundreds and 2 thousands remain: the total remainder is then 2248, and the work stands as below:—

Thousands.	Hundreds.	Tens.	Units.
4	6	7	9
2	4	3	1
2	2	4	8

It may be observed, that in this case the minuend and subtrahend are each regarded as four separate numbers, each column of units being separately subtracted independently of the others; and it matters not whether we begin from the thousands, and proceed from left to right, or begin from the units as usual, and proceed from right to left: we may even begin at the hundreds or tens, since the subtraction of any one order of units is quite independent of the others.

(105.) The same remainder would be obtained, with equal facility, if we sought the digits, which, added to those of the subtrahend, would produce the digits of the minuend: thus, it is obvious that 8 is the number which, added to 1, would produce 9; 4 added to 3 would produce 7; 2 added to 4 would produce 6, and 2 to 2 would produce 4. Thus the same remainder would be obtained by either method with the same ease and expedition, and might be obtained with equal facility in whatever order we should proceed, whether from

right to left, or from left to right. In the following example, however, we do not find the same facilities :—

			Units.
9	2	4	3
3	8	7	6

We cannot subtract 6 from 3, and therefore we borrow 1 from the 4 occupying the tens' place ; and, instead of subtracting 6 from 3, we subtract 6 from 13 : in the same manner, since we cannot subtract 7, the next figure from 3, which remains after 1 has been borrowed from 4, we borrow 1 from the 2 which fills the place of hundreds, and we subtract 7 tens from 13 tens. It becomes necessary, for the same reason, to transfer 1 from the thousands' place to the hundreds' place in the minuend. Instead of writing the minuend thus, in the ordinary way, 9243, we write it as below :—

Th	Hu		Units
8	11	13	13
3	8	7 _*	6
5	3	6	7

It will easily be perceived, that the two methods of ex-

pressing the number are equivalent to each other : subtracting 6 from 13, we now obtain a remainder 7 : subtracting 7 tens from 13 tens, we obtain a remainder 6 tens. In the same manner, subtracting 8 hundreds from 11 hundreds, we obtain a remainder 3 hundreds ; and subtracting 3 thousands from 8 thousands, we get the remainder 5 thousands.

(106.) Hence it may be perceived, that when any digit of the subtrahend is greater than the digit of the minuend immediately above it, we should add 10 to the latter, and subtract the former from the sum, to compensate for which it will be necessary to diminish by 1 the next figure of the minuend. It will be always possible to practise this except in the particular case when the next figure of the minuend is 0, as in the following example : —

9	8	0	2
5	7	4	8

The units of the minuend in this case being less than those of the subtrahend, it is necessary to borrow a unit from the tens of the minuend ; but the tens' place being occupied by 0, this is impossible : we may, therefore, borrow 1 from the hundreds ; taking 1, then, from the 8 hundreds, this 1 will be equivalent to 10 units and 9 tens : we add the 10 units to the 2 units in the units' place, and instead of 0 in the tens' place we substitute 9. The statement, therefore, takes the following form : —

Thousands.	Hundreds.	Tens.	Units.
9	7	9	12
5	7	4	8
4	0	5	4

The hundreds being diminished by 1, the tens replaced by 9, and 10 being added to the units, the subtraction will now be performed without difficulty, and we obtain the remainder 4054.

(107.) It appears, therefore, that when a figure of the minuend is less than that immediately below it in the subtrahend, and at the same time the preceding figure of the minuend is 0, then we must add 10 to the figure of the minuend which is less than that below it: conceive 9 to be substituted for the 0, and diminish by 1 the following figure of the minuend.

(108.) There is, however, an easier way of encountering this difficulty, derived from an observation already made, — that if the minuend and subtrahend be increased by the addition of the same number, no change will be made in the remainder. Giving due attention to this observation, let us suppose, in the above example, that we add 10 units to the units of the minuend. The effect of such an addition would be neutralised by adding, at the same time, 10 units to the units of the subtrahend. But as this would still leave the units of the subtrahend greater than those of the minuend, we make the same addition to the subtrahend by another method, viz. by adding 1 to its tens; so that the tens, instead of being 4, are increased by 1, and become 5. The statement of the question will then be as follows: —

hundreds.	hundreds.	ts.	ts.
9	8	10	12
5	8	5	8
4	0	5	4

We have here added 10 to the units of the minuend, and 1 to the tens of the subtrahend. These two additions neutralise each other, on the principle just referred to. We have also added 10 to the tens of the minuend, and at the same time added 1 to the hundreds of the subtrahend. These two additions in the same way neutralise each other. In fact, by the two additions which have been made, the minuend has been increased by the addition of 11 tens, or 110, and the subtrahend has been increased also by the addition of 110: the two additions, therefore, being equal, leave the remainder the same.

(109.) The following general rule will then serve for all cases in which it is required to subtract the lesser number from the greater:—

General Rule for Subtraction.

To subtract a lesser number from a greater, place the lesser number under the greater, so that the units of the same order, in the two numbers, shall stand in the same vertical column, the units under the units, the tens under the tens, and so on. Then subtract the units of the subtrahend from the units of the minuend, and write the remainder under them in the same column, placing, in the same way, the tens under the tens, and the hundreds under the hundreds, and so on, for every column, from right to left.

If it should happen that any figure of the minuend should be less than the figure of the subtrahend immediately below it, then increase the former by 10, and proceed as before ; but in that case it will be necessary to increase the next figure of the subtrahend by 1, or to carry 1 to it.

(110.) To be enabled to perform the operation of subtraction with ease and expedition by the above rule, it will be necessary that the student should retain in his memory the remainders which are found by subtracting the single digits from each other, and from all numbers between 10 and 19 inclusive ; but this knowledge he must needs possess if he has learned the table of addition given in the last chapter. Thus, if it be required to know what is left when 8 is subtracted from 17, the question is, what number added to 8 will produce 17. This, it will be perceived, forms a part of the table of addition.

(111.) The operation of subtraction is indicated by the sign $-$, *minus*, placed between the minuend and subtrahend. Thus, $7-2$ means that 2 is to be subtracted from 7 ; and it expresses the *remainder* which is obtained by that operation. Thus, $7-2=5$, means that 2 subtracted from 7 leaves the remainder 5.

(112.) Since no more than two numbers can be concerned in any question of subtraction, the operation is in general more simple than in questions of addition, where several numbers may be concerned. Subtraction, therefore, stands less in need of verification, being less liable to error ; but still it is desirable that the computer should possess some means of checking his work. An easy and obvious method of doing so is suggested by the fact, that the remainder, added to the subtrahend, must make a total equal to the minuend. When, therefore, any subtraction has been performed, add the subtrahend and remainder, and, if the sum be the same as the minuend, the work may be considered as correct.

(113.) The teacher may check the work of his pupils

without the labour of re-calculation, by giving to them, as already explained, questions in which the numbers are different, but which he knows must give the same remainder. He may always accomplish this by increasing any digit of the minuend by 1, and increasing the corresponding digit of the subtrahend also by 1, or increasing both by 2, or by any other number. This contrivance rests upon the principle already explained, that, when two numbers are equally increased their difference will remain the same. He may also increase any figure of the minuend alone, observing that the corresponding figure in the one remainder must be as much greater than the same figure in the other remainder. He may also increase any figure in the subtrahend, observing that the corresponding figure in the remainder must be equally diminished.

(114.) The following example will illustrate these observations:— Let it be required to subtract 3769 from 4354, and to verify the process.

$$\begin{array}{r}
 \text{M.} \quad - \quad - \quad 4354 \\
 \text{S.} \quad - \quad - \quad 3769 \\
 \hline
 \text{R} \quad - \quad - \quad 585 \\
 \hline
 \text{M.} \quad - \quad - \quad 4354 \\
 \hline
 \end{array}$$

We say 9 cannot be subtracted from 4, but, adding 10, and taking it from 14, we get the remainder 5. To compensate for the 10 added to the 4, we carry 1 to the 6; but 7 tens cannot be taken from 5 tens: we therefore add 10 to the 5, and, taking 7 tens from 15 tens, we set down the remainder, which is 8 tens. For the 10 added to the 5 above, we carry 1 to the 7 below, which makes it 8: 8 hundreds cannot be deducted from 3 hundreds; we therefore add 10 to the 3, and, deducting 8 from the 13, obtain the remainder 5 hundreds. For the 10 added to the 3 above, we carry 1 to the 3 below, which makes it 4: 4 thousands being deducted from the 4 thousands above, leaves no remainder. The

total remainder is, therefore, 585. To verify the process, this remainder is now added to the subtrahend, and we find that by such addition the minuend is reproduced: consequently the process is correct. If we wish to propose questions to different computers which will produce the same remainder, 585, we have only to increase or diminish equally the digits which occupy any vertical column in the minuend and subtrahend. Thus, if we add 1 to the hundreds and tens, the question will be stated and proved as follows: —

$$\begin{array}{r}
 \text{M} \quad - \quad - \quad 4464 \\
 \text{S} \quad - \quad - \quad 3879 \\
 \hline
 \text{R} \quad - \quad - \quad 585 \\
 \hline
 \text{M} \quad - \quad - \quad 4464 \\
 \hline
 \end{array}$$

In like manner, if the thousands and units be both diminished by 1, while the tens or hundreds are each increased by 2, the question will take the following form, with the same remainder: —

$$\begin{array}{r}
 \text{M} \quad - \quad - \quad 3573 \\
 \text{S} \quad - \quad - \quad 2988 \\
 \hline
 \text{R} \quad - \quad - \quad 585 \\
 \hline
 \text{M} \quad - \quad - \quad 3573 \\
 \hline
 \end{array}$$

If we increase each of the digits of the minuend by 1, the remainder will likewise have its digits increased by 1: —

$$\begin{array}{r}
 \text{M} \quad - \quad - \quad 5465 \\
 \text{S} \quad - \quad - \quad 3769 \\
 \hline
 \text{R} \quad - \quad - \quad 1696 \\
 \hline
 \text{M} \quad - \quad - \quad 5465 \\
 \hline
 \end{array}$$

In this case a 0 is understood to precede the 5 in the hundreds' place of the first remainder; and this 0 being increased by 1, brings 1 into the thousands' place in the above remainder. If each of the digits of the subtrahend be diminished by 1, the digits of the remainder will likewise be increased by 1, and the same change will be made as in the last case:—

$$\begin{array}{r}
 M \quad - \quad - \quad - \quad 4354 \\
 S \quad - \quad - \quad - \quad 2658 \\
 \hline
 R \quad - \quad - \quad - \quad 1696 \\
 \hline
 M \quad - \quad - \quad - \quad 4354 \\
 \hline
 \end{array}$$

It is unnecessary to pursue these examples further. Teachers will easily see how they may modify examples with unlimited variety, so as either to cause them all to have the same remainder, or to make any proposed change in one or more figures of the remainder, so that, by the correspondence of the results with these principles, they may be assured that the work of their pupils is correct without the labour of recalculation.

(115.) As addition becomes the means of verifying subtraction, so likewise we may use subtraction as a means of verifying addition.

Let several numbers, marked A, B, C, D and E, be added together by the rule for addition, so that the sum 3 shall be found:—

$$\begin{array}{r}
 A \quad - \quad - \quad - \quad 3579 \\
 \hline
 B \quad - \quad - \quad - \quad 2684 \\
 C \quad - \quad - \quad - \quad 3761 \\
 D \quad - \quad - \quad - \quad 2007 \\
 E \quad - \quad - \quad - \quad 9889 \\
 \hline
 S \quad - \quad - \quad - \quad 21920 \\
 S' \quad - \quad - \quad - \quad 18341 \\
 \hline
 A \quad - \quad - \quad - \quad 3579 \\
 \hline
 \end{array}$$

Let the top line A be now cut off, and let the four lines B, C, D and E be added ; write their sum, S', under the total sum S. It is evident that since S' is the sum of all the proposed numbers except A, we ought to get the number A if we subtract S' from S ; and we accordingly find that the remainder of such subtraction is the number A. Had it been otherwise, the inference would be, that the addition was performed incorrectly, and required revision.

(116.) The principle of this method of verification may be extended and varied, so as to afford exercise to the judgment and understanding of the pupils. After the addition has been performed, let the master strike one figure out of each vertical column, and direct the pupil to add the numbers, omitting the figures thus crossed out. The sum thus obtained being subtracted from the total sum, the remainder should be a number consisting of the same figures as were struck out of the numbers added. The following example will illustrate this : —

A	-	-	1857
B	-	-	8642
C	-		7684
D	-	-	9876
E	-	-	7530
		•	<hr/>
S	-	-	35059
S'	-		26703
			<hr/>
R	-	•	8356
			<hr/>

Here the numbers A, B, C, D and E, being added together, produce the total S. A figure is then struck out of each vertical column, and the numbers are again added, omitting those figures. The sum S' is thus obtained. S' being then subtracted from S, we obtain a remainder composed of the same figures as were struck out of the original numbers, and occurring in the same places. The reason of this is sufficiently obvious : by

striking out 8 in the number B, 3 in A, 5 in C, and 6 in D, we diminished the amount of those numbers respectively, — viz. that of B by 8000, that of A by 300, that of C by 50, and that of D by 6. The total amount of the whole was, therefore, diminished by 8356: consequently S', which is the sum of what remained, ought to be less than S, the total sum, by 8356, which is accordingly the remainder.

(117.) Another mode of verifying the results of subtraction, and indeed also of addition, is by proceeding from left to right, instead of from right to left. This, although not the most convenient in practice, yet should be occasionally resorted to by teachers, with a view to familiarise the pupils with the reasons of the arithmetical processes. In the following number the subtrahend has been deducted from the minuend first, in the usual way, by proceeding from right to left, and thus the remainder has been found. It may be subsequently verified by the following process: —

$$\begin{array}{r}
 \text{M} \quad - \quad - \quad - \quad 5641 \\
 \text{S} \quad - \quad - \quad - \quad 3760 \\
 \hline
 \text{R} \quad - \quad - \quad - \quad 1881 \\
 \hline
 \end{array}$$

Commencing from the left; we shall subtract from the minuend both the remainder and the subtrahend. As these two numbers together must be equal to the minuend, the remainder in this subtraction ought to be nothing. Subtracting, then, 3 thousands from 5 thousands, we get the remainder, 2 thousands; and subtracting 1 thousand from this, 1 thousand remains; this being added to the next figure, 6, of the minuend, makes 16 hundreds. Subtracting from this the 7 hundreds of the subtrahend, there remain 9 hundreds; and from this subtracting the 8 hundreds of the remainder, there remains 1 hundred; this being added to the succeeding 4 of the minuend, gives 14 tens. The 6 tens of the subtrahend being taken from this, leaves

8 tens ; and the 8 tens of the remainder being taken from this, leaves nothing. The 1 in the units' place of the remainder being then taken from the 1 in the units' place of the minuend, leaves 0. Consequently the subtrahend and remainder, being successively subtracted from the minuend, leave no remainder.

(118.) Upon the same principle questions in addition may be verified. In the following example the numbers are first added by proceeding from right to left in the usual way : —

$$\begin{array}{r}
 A \quad - \quad - \quad 376 \\
 B \quad - \quad - \quad 489 \\
 C \quad - \quad - \quad 768 \\
 \hline
 S \quad - \quad - \quad 1633 \\
 \hline
 \quad \quad \quad \not{470} \\
 \hline
 \end{array}$$

To verify the work by proceeding from left to right we add the first column, and find 14 hundreds ; subtracting this from the 16 hundreds in the sum, 2 hundreds are left : we write therefore 2 under the 6. Adding the tens' column we find the sum 21 ; this being taken from the 3 tens of the sum and the 2 hundreds which remained, or, what is the same, from 23 tens, 2 tens remain : we therefore strike out the 2 under the 6, and write 2 in the tens' place under the 3. The units' column being now added gives 23 ; but there remain of the sum 2 tens and 3 units, from which the 23 obtained by the addition of the units' column being taken, there is no remainder : thus the successive columns being added, from left to right, and subtracted from the total, leave no remainder, and the work is therefore correct.

CHAP. V.

MULTIPLICATION.

BEFORE we enter into the details of the arithmetical operations of multiplication and division, it will be useful to explain some properties by which certain classes of number are distinguished, relatively to their composition by the addition of other numbers.

(119.) We have seen that all numbers whatever may be composed by the addition of units. Thus the composition of the successive numbers, 1, 2, 3, &c. may be exhibited in the following manner :—

$$\begin{aligned} 1 &= 1 \\ 1 + 1 &= 2 \\ 1 + 1 + 1 &= 3 \\ 1 + 1 + 1 + 1 &= 4 \\ &\&c. \&c. \end{aligned}$$

This property of being formed by the continued addition of the number 1, is common to all numbers whatever ; but certain numbers are distinguished by a like property with respect to others ;—that is to say, of being formed by the continued addition of other numbers, such as 2, 3, 4, &c. The numbers which are formed by the continued addition of 2 are as follows :—

$$\begin{aligned} 2 &= 2 \\ 2 + 2 &= 4 \\ 2 + 2 + 2 &= 6 \\ 2 + 2 + 2 + 2 &= 8 \\ &\&c. \&c. \end{aligned}$$

In like manner, the numbers formed by the continued addition of 3, 4, &c. are expressed as follows :—

$$3=3$$

$$3+3=6$$

$$3+3+3=9$$

$$3+3+3+3=12$$

&c. &c. &c.

$$4=4$$

$$4+4=8$$

$$4+4+4=12$$

$$4+4+4+4=16$$

&c. &c. &c.

It will be perceived that in the series of numbers proceeding upwards from 1, some will be found which cannot be formed by the continued addition of any other number except 1; the others may be formed by the continued addition of 2, 3, or some other higher number. Thus the numbers 3, 5 and 7 cannot be produced by the continued addition of any other number, 1 excepted, while the intermediate numbers 4 and 6 admit of being formed, the first by the addition of 2, and the second by the addition of either 2 or 3.

(120.) Those numbers which cannot be formed by the continued addition of any number except 1, are distinguished by the name of *prime numbers*: thus the numbers 11, 13, 17, &c. are prime numbers.

(121.) All other numbers are called *multiple numbers*, and they are said to be *multiples* of those lesser numbers by the continued addition of which they may be formed: thus 6 is a *multiple* of 2, because it may be formed by the addition of 3 twos. It is likewise a *multiple* of 3, because it may be formed by the addition of 2 threes. In the same manner 12 is a multiple of 2, of 3, of 4, and of 6, because it may be formed by the addition of 6 twos, 4 threes, 3 fours, or 2 sixes.

(122.) In the ascending series of numbers every alternate number is a multiple of two. This will easily be perceived when it is considered that each successive number in the series is formed by adding 1 to the preceding number: beginning then at the second number

of the series, namely 2 itself, we proceed to the next but one by adding 2, and pass to the next alternate number by adding 2 more, and so on. From the very nature of the series, therefore, every alternate number beginning from the second is formed by the constant addition of 2, and is therefore a multiple of 2. Such numbers are called *even numbers*, and the intermediate numbers of the series are called *odd numbers*: thus the series of even numbers are 2, 4, 6, 8, 10, &c., and the series of odd numbers 1, 3, 5, 7, 9, &c.

(123.) Since every even number is a multiple of 2, it is evident that no even number, except 2 itself, can be *prime*, and therefore *every prime number, except 2 itself, must be an odd number*. We must not, however, infer on the other hand that every odd number is prime: 9 is a multiple of 3, 15 of 5, &c.

(124.) Since in the series of numbers an odd number always stands between two even ones, it follows that an even number will be obtained either by subtracting one from an odd number, or adding 1 to it.

(125.) In like manner, since an even number always stands between two odd ones, we shall get an odd number either by subtracting 1 from an even number, or adding 1 to it.

(126.) Two numbers are said to be the *same multiples* of two others, when they are formed by the addition of those two others the same number of times: thus 12 being formed by the addition of 4 threes, and 20 by the addition of 4 fives, 12 is the same multiple of 3 as 20 is of 5.

(127.) The number, by the continued addition of which another is formed, is called a *sub-multiple* of that other: thus 12 being formed by the continued addition of 2, 3, 4 or 6, these numbers are severally *sub-multiples* of 12. In fact, if one number be a *multiple* of another, then the latter must always be a *sub-multiple* of the former.

(128.) When a number is not a multiple of another number, it is frequently necessary to consider the two successive multiples of that other between which it is

placed in the numerical series, and "to mark its distance from the one or the other of these. Let us suppose that it is required to determine whether 40 be a multiple of 9, and if not, to determine how it stands with respect to the two successive multiples of 9, between which it is placed in the numerical series. By adding 4 nines and 5 nines respectively we obtain the following results,

$$\begin{aligned} 9 + 9 + 9 + 9 &= 36 \\ 9 + 9 + 9 + 9 + 9 &= 45 \end{aligned}$$

It appears, then, that 40, the number in question, is greater than that which is produced by the addition of 4 nines, and less than that which results from the addition of 5 nines. The composition of the number 40, so far as it can be made by the addition of nines, would then be expressed in this manner,

$$9 + 9 + 9 + 9 + 4 = 40$$

The greatest multiple of 9 contained in 40 is the sum of 4 nines, and the number 40 exceeds this by 4.

In the same manner, if we enquire how far the composition of the number 53 can be effected by the continued addition of 8, we shall find the following results,

$$\begin{aligned} 8 + 8 + 8 + 8 + 8 + 8 &= 48 \\ 8 + 8 + 8 + 8 + 8 + 8 + 8 &= 56. \end{aligned}$$

Now it appears that the number 53 is placed in the numerical series between the two multiples of 8, formed by the addition of 6 eights and 7 eights. The composition of the number 53 is then expressed as follows,

$$8 + 8 + 8 + 8 + 8 + 8 + 5 = 53$$

(129.) When it is necessary then to investigate the composition of any number by the continued addition of any other number, the result of the investigation will always be, either that the former is a multiple of the latter,—in which case it will be necessary to state what multiple it is,—or that the former exceeds a certain multiple of the latter, by a number which is always less than the latter number. In fact, we must always de-

clare *what multiple the number is, if it be a multiple, or if not we must determine the greatest multiple of the proposed number which it contains, and its excess above that multiple.*

It is necessary that these points should be distinctly understood by all who expect to form clear and distinct conceptions of the processes of MULTIPLICATION and DIVISION.

(130.) The object of multiplication is to furnish a method of discovering the number which would be produced by adding the same number, any proposed number of times, more concise and expeditious than the ordinary methods of addition already explained. *Multiplication, is, therefore, that arithmetical operation by which we may with brevity and facility discover any proposed multiple of any proposed number.*

(131.) The data necessary in such a question are the number whose multiple is to be found, and the number of times that number is to be repeated in order to form the proposed multiple.

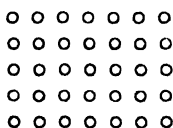
The number by the continued addition of which the sought multiple is formed, is called the *multiplicand*, or the number *to be multiplied*; the number of times which that number must be repeated in order to form the sought multiple is called the *multiplier*, and in this case the result of the operation, which is the multiple sought, is called the *product*.

Thus if it be required to discover what number would be produced by the repetition of 9 six times, then 9 is the *multiplicand*, and 6 the *multiplier*. If we proceeded by the method of addition, we should write down 9 in a column 6 times, and we should add the column by the ordinary rules of addition, and should find that the addition would produce 54; this number 54 would then be the *product*.

The brevity and expedition which are obtained by the methods of multiplication commonly practised, depend partly on the student committing to memory the results of certain simple multiplications, so as to apply them

when necessary, and partly on certain properties of number, which we shall now proceed to explain.

(132.) If we add together 5 sevens, we shall find that their sum is 35 : now, if we add together 7 fives, we shall find that their sum is the same. In the one case 7 is the multiplicand, and 5 the multiplier ; in the other 5 is the multiplicand and 7 the multiplier. In this case, therefore, it appears that *the product will be the same if the multiplicand be changed into the multiplier, and the multiplier into the multiplicand*. This is a general property of all numbers, and it will be found universally that when any number multiplied by another gives a certain product, the same product will be obtained if the latter be multiplied by the former. The reason of this will become apparent if we suppose the numbers in question represented by counters. Taking the example already given, let us express by counters 5 times 7 ; we shall place 7 counters in a row, and repeat that row 5 times : the arrangement will then be as follows ;—



We have here five cross rows of 7, and, therefore, the total number of counters is 5 times 7 ; but if we consider the same collection of counters in another point of view, we shall see that they are also 7 times 5. It will be perceived that the collection consists of 7 upright columns, each column containing 5 counters : the total number is therefore 7 times 5.

The same illustration will be applicable to any multiplier and multiplicand, and will therefore establish the general conclusion, that the multiplier and multiplicand may be interchanged without affecting the product.

(133.) Since there is no real distinction, then, between the multiplier and multiplicand in their relation to the product, there is no reason why they should be called by

different names, and accordingly they are sometimes denominated by the common name *factors*. Thus 48 is a *product* whose *factors* are 6 and 8, or 4 and 12, &c.

(134.) A number may be multiplied by another by resolving it into several parts, multiplying each of those parts separately, and then adding the products. This will become evident when we apply it to an example. Let us suppose that we wish to multiply 6 by 4; 6 may be formed by the addition of 1, 2, and 3, so that

$$1 + 2 + 3 = 6$$

To multiply 6 by 4 we should add together 4 sixes: following the simple method of addition, the process would be as follows,

$$\begin{array}{r}
 1 + 2 + 3 = 6 \\
 1 + 2 + 3 = 6 \\
 1 + 2 + 3 = 6 \\
 1 + 2 + 3 = 6 \\
 \hline
 4 + 8 + 12 = 24
 \end{array}$$

Here we have obtained the product in two distinct ways, either by adding the 4 sixes, or by adding separately the three columns in which 1, 2 and 3 are four times repeated. In the latter case we obtain the 3 products, 4, 8 and 12, and by the addition of these three we must, of course, get the same result as we have obtained by adding the column of four sixes. The reader will find no difficulty in generalising this result, and he will perceive that, universally, *the multiplicand may be resolved into parts, each part being multiplied by the multiplier, and the total product will be obtained by adding together these partial products.*

(135.) It is not difficult to show, that the principle just explained involves another proof that the multiplier and multiplicand are interchangeable. Let us suppose that we are required to multiply 8 by 6: we may resolve the multiplicand 8 into eight component units, so that we should have its composition expressed thus:—

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 8$$

Now, by the principle just established, we shall multiply 8 by 6, providing that we multiply each of the component parts of 8 by 6, and add together the products; but if we add these we shall obtain 8 sixes, since each unit, multiplied by 6, will give the product 6: thus we perceive that 6 eights are equivalent to 8 sixes.

(136.) As we have already proved that the multiplier and multiplicand are interchangeable without affecting the product, it follows from what has been just established, that multiplication may be effected by resolving the multiplier into smaller numbers, and multiplying the multiplicand by each of those numbers separately. Thus, if the multiplier be 8, we may perform the multiplication by multiplying the multiplicand by 5 and 3 separately, and adding together the products. Multiplication may, therefore, be always performed either by resolving the multiplicand into smaller numbers, and multiplying each of these by the multiplier, or resolving the multiplier into smaller numbers, and multiplying the multiplicand by each of these. In either case the several partial products being added together, the total product will be the result.

(137.) The principle just explained facilitates the process of multiplication by making the multiplication of large numbers depend on that of small ones; but still more by a circumstance which will be explained more fully hereafter, namely, that there are some numbers whose multiplication involves no difficulty beyond a knowledge of the notation of number. We have already seen that a number may be multiplied by 10, 100, 1000, &c. by merely annexing noughts to it.

(138.) If the multiplier be not a prime number (120.), it will always be itself the product of two other numbers. In such a case the multiplication may be performed by using these two numbers successively as multipliers, instead of the given multiplier. Thus, if the given multiplier be 6, we may first multiply by 2,

and then multiply the product thus found by 3: the final product will be the same as if we had multiplied in the first instance by 6. To understand this process, it is only necessary to refer to an example. Let us suppose that we wish to multiply 8 by 6: by what has been just established, if we resolve 6 into 3 parts, each of which shall be 2, the multiplication will be effected by multiplying 8 by its three parts severally, and then adding the products, so that in the final result we shall have the product of 8 and 2 three times repeated; that is, we shall have that product multiplied by 3. It is evident, then, that the final product is 3 times the product of 8 and 2. There can be no difficulty in perceiving that this reasoning is universally applicable.

(139.) In order to multiply with ease and despatch any numbers exceeding 9, it is necessary that the product of every pair of single digits should be first investigated and committed to memory. The products of the smaller digits can only be found by addition. Those of the greater ones may be determined by combining addition with the multiplication of inferior digits by some of the principles already explained. We shall presently explain these more fully; meanwhile we shall exhibit in the following table the product of every pair of single digits.

The operation of multiplication is expressed by the sign \times placed between the two numbers to be multiplied: thus $2 \times 3 = 6$ means that the product of 2 and 3 is 6:—

6
MULTIPLICATION TABLE*

$2 \times 2 = 4$	$3 \times 3 = 9$	$4 \times 4 = 16$	$5 \times 5 = 25$	$6 \times 6 = 36$	$7 \times 7 = 49$	$8 \times 8 = 64$	$9 \times 9 = 81$
$2 \times 3 = 6$	$3 \times 4 = 12$	$4 \times 5 = 20$	$5 \times 6 = 30$	$6 \times 7 = 42$	$7 \times 8 = 56$	$8 \times 9 = 72$	
$2 \times 4 = 8$	$3 \times 5 = 15$	$4 \times 6 = 24$	$5 \times 7 = 35$	$6 \times 8 = 48$	$7 \times 9 = 63$		
$2 \times 5 = 10$	$3 \times 6 = 18$	$4 \times 7 = 28$	$5 \times 8 = 40$	$6 \times 9 = 54$			
$2 \times 6 = 12$	$3 \times 7 = 21$	$4 \times 8 = 32$	$5 \times 9 = 45$				
$2 \times 7 = 14$	$3 \times 8 = 24$	$4 \times 9 = 36$					
$2 \times 8 = 16$	$3 \times 9 = 27$						
$2 \times 9 = 18$							

* As the multiplication table is most commonly given it is twice the length of the above: this arises from the circumstance of the product on

The same results may be exhibited in a still more convenient form as follows: —

	2	3	4	5	6	7	8	9
9	18	27	36	45	54	63	72	81
8	16	24	32	40	48	56	64	
7	14	21	28	35	42	49		
6	12	18	24	30	36			
5	10	15	20	25				
4	8	12	16					
3	6	9						
2	4							

To find the product of two numbers by this table, we must look for the greater number in the first upright column on the left, and for the lesser in the highest cross row. The product of the two numbers will be found in the same cross row with the greater number, and in the same upright column with the less. Thus, if we wish to find the product of 8 and 5, we look along the cross row from 8, until we find the number which is directly under 5: this number, 40, is the product of 8 and 5.

(140.) All the products contained in this table may be found directly by addition, and, as the numbers to be added cannot, in any case, exceed 9, such a process of

the same pair of numbers being given in two different forms; thus it is usual to insert "twice 3 make 6," and also "3 times 2 make 6." The pupil, however, having once understood that the multiplicand and multiplier are interchangeable, this repetition becomes quite unnecessary, and the greater length of the table has a tendency only to confuse the pupil, and needlessly to burthen his memory.

addition will not in any case be attended with labour or difficulty. Nevertheless, by bearing in mind some of the principles respecting multiplication which have been already established, the larger products contained in the multiplication table may be easily inferred from the smaller ones. If we wish to prove, for example, that the product of 9 and 8 is 72, we may resolve 9 into 4 and 5 (134.), and multiply each of these by 8. We find, by the other parts of the table, that the product of 8 and 4 is 32, and that the product of 8 and 5 is 40: these numbers added together obviously make 72, which is therefore the product of 8 and 9.

Again, if we wish to prove that the product of 6 and 7 is 42, we may resolve 7 into the parts 2, 3, and 2: these three severally multiplied by 6 give 12, 18, and 12, which added together make 42. In the same manner all the higher products in the table may be inferred from the lower ones.

Having committed to memory the products contained in the above table, the student will be prepared to practise and comprehend the process of multiplication in those cases in which one or both of the numbers multiplied consist of two or more digits. In these cases, however, the more complex questions require a previous knowledge of the methods of solution for the more simple ones. In unfolding the principles on which the general rule for the multiplication of any numbers depends, it will be necessary that we should proceed step by step, from the most simple questions which can be proposed, through intermediate degrees of complexity, in order to arrive at a general method.

(141.) We shall first consider the case in which one of the factors is a single digit, and the other a number consisting of several places. As it is a matter of indifference which of the two factors is considered as the multiplier (132.), we shall, for convenience, consider the latter number as the multiplicand, and the former as the multiplier. Let the multiplicand, then, be 23789, and the multiplier 6. The question, if solved by the

direct method of addition, would be worked as follows : —

$$\begin{array}{r}
 23789 \\
 23789 \\
 23789 \\
 23789 \\
 23789 \\
 23789 \\
 \hline
 142734
 \end{array}$$

Having, however, previously committed to memory the multiplication table, this process of addition admits of being considerably expedited. Instead of adding together the six nines in the units column, we recall that 6 times 9 make 54, which being the sum of the units column, we write down 4 in the units place of the sum, and carry 5 to the tens column by the rule for addition (93.). In like manner, instead of adding the numbers in the tens column, we learn from the multiplication table that 6 eights are 48, to which adding the 5 carried, we get 53: by the rule for addition, we put 3 in the tens place, and carry 5 to the hundreds. We proceed in the same way, finding the sum of each column, not by addition, but by the multiplication table, adding, however, to the result, thus found, the number carried from the preceding column. In this process, it will be perceived that we depart in no respect from the practice established in addition, except merely in our way of obtaining the knowledge of the number which would be found by adding the figures of each column. It will, therefore, be evident that such a process is nothing, but addition somewhat abridged, or rather expedited. The figures of every column being the same, it is not necessary that the numbers should be written one under another, provided that we have any way of expressing the number of times which each figure occurs in the upright columns. In the present case, the units column consists of 6 nines, the tens column of 6 eights, the hundreds column of 6 sevens, and so on. The process,

may, therefore, be expressed without repeating the multiplicand, as follows : —

$$\begin{array}{r} 23789 \\ 6 \\ \hline 142734 \end{array}$$

In this case the multiplier, 6, written under the units place of the multiplicand, signifies the number of times the multiplicand is understood to be repeated ; and it will be evident that the process of multiplication will consist in multiplying each digit of the multiplicand, beginning from the units place and proceeding from right to left, the product of the multiplier by each digit of the multiplicand to be treated in the same manner as the number found by adding the successive columns in addition.

The *rationale* of the process should be strongly impressed upon the mind of the young pupil by every teacher, and this may be always effected by causing the pupil, as he works each question, to repeat the results of his work in the following manner : —

“ Six nines are 54, or 5 tens and 4 units : I write 4 in the units place, and I reserve 5 to be added to the tens. Six eights are 48, to which the 5 tens carried being added, make 53 tens, or, what is the same, 5 hundreds and 3 tens : in the tens place I write 3, and reserve the 5 hundreds to be added to the hundreds. Six sevens are 42, which being hundreds are 42 hundreds ; adding the 5 hundreds carried, I obtain 47 hundreds, or, what is the same, 4 thousands and 7 hundreds : in the hundreds place I write 7, and reserve the 4 thousands to be added to the thousands. Six threes are 18, which being in the thousands place, are 18 thousands ; adding the 4 carried, we have 22 thousands, or 2 ten thousands and 2 thousands : I write 2 in the thousands place, and reserve 2 for the ten thousands. Six twos are 12, which being in the ten thousands place, are 12 ten thousands ; adding to this the 2 carried, I have 14 ten thousands, or 1 hun-

dred thousand, and 4 ten thousands: I write 4 in the ten thousands place, and 1 in the place of hundred thousands."

When the pupil has been made to work a sufficient number of examples in this manner, he will distinctly perceive and remember the reasons for the several steps of the process. It may then be useful to cause him to commit to memory the following rule, which is nothing more than a statement in general terms of what he will have already applied in the various particular examples: —

Rule.

(142.) When the multiplier is a single digit, the product is found by multiplying the several digits of the multiplicand by the multiplier, proceeding from right to left. The figures of the product are to be written as in addition, and the numbers carried from place to place, determined in the same way.

(143.) It is evident from what has been explained above, that if the multiplicand end in one or more ciphers, the product will also terminate in ciphers, because, if the multiplicand be obtained by the direct method of addition, the columns of ciphers when added will give a cipher for the same place in the product. If we would, therefore, multiply such a number as 23000 by 5, the operation would be as follows: —

$$\begin{array}{r} 23000 \\ \quad 5 \\ \hline 115000 \end{array}$$

This process may often be abridged by omitting the ciphers of the multiplicand in the first instance, multiplying the other figures only, the ciphers omitted being subsequently annexed to the product. Thus, in the example just given, we should multiply 23 by 5, and should obtain the product 115. Having done this, to get the final product we have only to annex three noughts. When the multiplicand is a high number, terminating

in a great number of ciphers, the process is materially abridged by this means.

(144.) Let us next consider the case in which the multiplier is a number consisting of two places, the latter place being filled by a cipher : thus, let it be required to multiply 23789 by 60. It has been already shown, that we can multiply a number by 60 by first multiplying it by 6, and then multiplying the product by 10. (138.) But a number is multiplied by 10 by merely annexing to it a cipher : hence, to multiply by 60 we multiply by 6, and annex a cipher to the product : the process is as follows : —

$$\begin{array}{r} 23789 \\ 60 \\ \hline 1427340 \end{array}$$

Hence it appears that when the multiplier is a number consisting of a significant digit followed by a cipher, the operation is performed with as much ease and expedition as if the multiplier were a single digit. The same will be found to be true when the multiplier is a single digit followed by two or more ciphers. Let us take the same example, the multiplier being 600 : the multiplication may be performed, in this case, by first multiplying by 6, and then multiplying the product by 100. (138.) But since a number is multiplied by 100 by annexing two ciphers to it (65.), the multiplication by 600 is performed by multiplying by 6, and annexing two ciphers to the product. The operation, in the proposed example, is as follows : —

$$\begin{array}{r} 23789 \\ 600 \\ \hline 14273400 \end{array}$$

If the multiplier be 6000, it may be shown, by the same reasoning, that the product will be found by multiplying by 6, and annexing three noughts ; and the same for multipliers ending in a greater number of noughts.

(145.) These principles being clearly fixed in the mind of the pupil, he will be prepared to understand the details of the more complex questions in multiplication. Let us take the case where the multiplier consists of two significant digits: retaining the same multiplicand, let the multiplier be 67. We have seen (136.) that multiplication may be performed by resolving the multiplier into two parts, multiplying the multiplicand by each of these parts separately, and adding together the products thus found. In the present case, let us conceive the number 67 resolved into two parts, 60 and 7: we shall multiply the multiplicand by each of these parts separately.

$$\begin{array}{r}
 \text{Multiply} \quad - \quad - \quad - \quad 23789 \\
 \text{by} \quad - \quad - \quad - \quad - \quad 7 \\
 \hline
 \text{Product} \quad - \quad - \quad - \quad 166523 \\
 \text{Multiply} \quad - \quad - \quad - \quad 23789 \\
 \text{by} \quad - \quad - \quad - \quad - \quad 60 \\
 \hline
 \text{Product} \quad - \quad - \quad 1427340 \\
 \text{Add} \quad - \quad - \quad - \quad 166523 \\
 \text{to} \quad - \quad - \quad - \quad 1427340 \\
 \hline
 \text{Total product} \quad - \quad 1593863
 \end{array}$$

Such is the product of the two numbers required; but the process may be written in a much more abridged form. Instead of actually resolving the multiplier into two parts, we shall *imagine* it to be so resolved, and shall perform the multiplication as if it were so: the process which has just been written at length may be expressed more shortly in the following manner: —

$$\begin{array}{r}
 23789 \\
 67 \\
 \hline
 166523 \\
 1427340 \\
 \hline
 1593863
 \end{array}$$

(146.) Let us next suppose the multiplier to consist of three digits, such as 673. In this case we shall conceive it resolved into three parts, 600, 70, and 3 : by each of these it is necessary to multiply the multiplicand, and when the products are obtained, they must be added together to get the total product. The process, stated at full length, would be as follows : —

Multiply	-	-	23789
by	-		3
Product	-	-	<u>71367</u>

Multiply	-	-	-	23789
by	-	-	-	70
Product	-			<u>1665230</u>

Multiply	-	-	23789
by	-	-	600
Product			<u>14273400</u>

Add - -	{	71367
		1665230
		14273400
Total product -		<u>16009997</u>

As in the former case, this process may be abridged as follows:—

				23789
				673
A	-	-	-	71367
B	-	-	-	1665230
C	-	-	-	14273400
P	-	-	-	16009997

In this case, the first line, A, is the partial product found by multiplying the multiplicand by 3; the line B is

found by multiplying the multiplicand by 70 ; and the line C by multiplying it by 600. The total product, P, is found by adding together these partial products. To find the line B, it is only necessary to place a nought under the units place of the first product A, and then multiply the multiplicand by 7, placing the successive figures found under the succeeding places of the product A, proceeding from right to left. It is evident that this is equivalent to multiplying the multiplicand by 7 and annexing a nought to the product, or, in other words, to multiplying the multiplicand by 70. In the same manner, by writing noughts in the units and tens place of the third product, and then multiplying the multiplicand by 6, we in fact multiply by 6, and annex two noughts to the product, which is equivalent to multiplying by 600.

But, with the slightest attention, it must be apparent that the introduction of the noughts into the products B and C, is of no other use than to throw the figures of these partial products so many places to the left. The nought which terminates the product B has no other effect than to remove the figure 3 from the units column to the tens column, and every other figure in the same product one place farther to the left. In the same manner, the addition of the two noughts to the product C has the effect only of removing each of the figures two places to the left, by which the units figure is transferred to the hundreds column, the tens to the thousands, and so on. That the ciphers which occur in the operation, as written above, have no other effect than those just mentioned, will be quite apparent if it be considered that no effect will be produced upon the total product P, by expunging the ciphers annexed to the partial products B and C, *provided that the other figures of B and C are allowed to retain the places assigned to them in the above method of writing the process.* If the ciphers were omitted, subject to this condition, the operation would be written in the following manner : —

$$\begin{array}{r}
 23789 \\
 673 \\
 \hline
 71367 \\
 166523 \\
 142734 \\
 \hline
 16009997
 \end{array}$$

(147.) Let us take another example. Suppose it required to multiply 65379 by 47853. Placing the multiplier under the multiplicand, as before, we shall consider the former as consisting of five distinct parts, expressed by the local values of its several digits ; these parts will be the following : —

$$\begin{array}{r}
 3 \\
 50 \\
 800 \\
 7000 \\
 40000
 \end{array}$$

and by these, successively, we must multiply the multiplicand. When the five partial products are obtained, the total product will be found by adding them together : the process, expressed at length, is as follows :—

$$\begin{array}{r}
 65379 \\
 47853 \\
 \hline
 A 196137 \\
 B 3268950 \\
 C 52303200 \\
 D 457653000 \\
 E 2615160000 \\
 P 3128581287 \\
 \hline
 \end{array}$$

In the line A we have the product of the multiplicand multiplied by 3 : the line B is found by multiplying the multiplicand by 5, and annexing a nought to the product, which is in effect multiplying it by 50 ; and, consequently, the number B is the second partial product. In the same manner the number C is found by multi-

plying the multiplicand by 8, and annexing two ciphers to the product, which is equivalent to multiplying it by 800 : the number C is, therefore, the third partial product. In the same way it may be shown that the numbers D and E are the two remaining partial products : these five numbers, being added together, give the total product, P.

As in the former example, it must be apparent that, provided the *places* of the significant digits in the products B, C, D, and E are preserved, the presence of the ciphers produces no effect in the addition by which the total product P is obtained. It is, therefore, unnecessary to write these ciphers in the process ; but, in omitting them, care must be taken *to maintain the other figures in those places which they would have if the ciphers were inserted.*

As the ciphers, then, produce no effect on the total product, it is usual to omit them, and to write down the process thus : —

$$\begin{array}{r}
 65379 \\
 47853 \\
 \hline
 A \dots\dots 196137 \\
 B \dots\dots 326895 \\
 C \dots\dots 523032 \\
 D \dots\dots 457653 \\
 E \dots\dots 261516 \\
 \hline
 P \dots\dots 3128581287
 \end{array}$$

(148.) The principle already explained in the case where the multiplier is a single digit, followed by one or more ciphers, will be equally applicable where the multiplier is a number consisting of several places terminating in one or more ciphers. In that case, the final ciphers of the multiplier may be omitted in the first instance : and after the multiplication has been thus performed, the same number of ciphers should be annexed to the product. The proof of this is precisely the same as the proof given in the case where the multiplier was a single digit followed by ciphers. Let us

suppose that the multiplier is 47000 ; in that case, the multiplier being the product of 47 and 1000, we shall obtain the true product by first multiplying by 47, and then multiplying the result by 1000. (138.) But the latter operation is performed by merely annexing three noughts to the product first obtained. (65.) To multiply 65379 by 47000, the process would then be as follows : —

$$\begin{array}{r}
 65379 \\
 \times 47 \\
 \hline
 457653 \\
 261516 \\
 \hline
 3072813 \\
 \hline
 3072813000
 \end{array}$$

(149.) From what has been just proved, combined with what was formerly proved in (65.) it will follow, that, when the multiplicand and multiplier both terminate in one or more ciphers, the multiplication may be performed by omitting the ciphers altogether in the first instance, and annexing to the number which results from the operation as many ciphers as were omitted in the multiplicand and multiplier taken together. Thus, if the multiplicand terminated in three ciphers, and the multiplier in two, we should annex five to the product. For example, let it be required to multiply 65000 by 3300: the process would be as follows : —

$$\begin{array}{r}
 65 \\
 \times 33 \\
 \hline
 195 \\
 195 \\
 \hline
 2145 \\
 \hline
 214500000
 \end{array}$$

(150.) If one or more figures of the multiplier happen to be ciphers, these figures not being final, the process

is somewhat modified ; but the principles on which it rests are the same. In the example already given, let us suppose that the 8 is removed from the hundreds place, and replaced by a nought : it is plain that in that case the third partial product, C, which was obtained by multiplying the multiplicand by 800, can have no place in the process while the other 4 partial products remain unchanged. The operation will, therefore, stand as before, with the exception that the product C will be omitted, and the process will be expressed as follows : —

$$\begin{array}{r}
 65379 \\
 47053 \\
 \hline
 A \dots\dots 196137 \\
 B \dots\dots 326845 \\
 D \dots 457653 \\
 E \dots 261516 \\
 \hline
 3076278087
 \end{array}$$

It will be observed here, that the units figure of the third partial product, D, is not placed under the tens of the product B, as it would be in ordinary cases, but is placed under the hundreds figure of the product B, and under the thousands figure of the product A. The reason of this will be easily understood : the partial product D is obtained by multiplying the multiplicand by 7, and annexing three noughts to the result ; these three noughts, if inserted, would stand under the last three figures of the product A, and consequently the figure 3 would stand in the thousands column of the numbers to be added. When a nought occurs, therefore, in the multiplier, it should be omitted in the multiplication ; but *the units' figure of the following partial product must be placed under the hundreds figure of the preceding one instead of the tens.*

(151.) Let us take the case where two or more noughts occur in succession in the multiplier, not occupying the final places. Let the multiplier, for example, be 40003 : the process will be as follows : —

$$\begin{array}{r}
 65379 \\
 40003 \\
 \hline
 A \dots\dots 196137 \\
 E \dots 261516 \\
 \hline
 2615356137
 \end{array}$$

In this case, the partial products, B, C, and D, disappear, and the figures of the product E maintain the same places, with respect to those of A, as they had when the intermediate products existed. The reason of this will be quite apparent, when it is considered that the product E is understood to be followed by four ciphers, which are merely omitted for the sake of brevity. These four ciphers would stand under the last four places of the partial product A.

(152.) From all that has been now explained we, may derive the following general rule for multiplication.

GENERAL RULE.

I. *Place the multiplier under the multiplicand, as in Addition.*

(It will be convenient always to consider the smaller of the two numbers to be multiplied together as the multiplier, and the greater as the multiplicand.)

II. *Multiply the multiplicand separately by every significant digit which is found in the multiplier, by which you will obtain as many partial products as there are significant digits in the multiplier.*

III. *Write these products one under the other, so that the last figure of each shall be under that figure of the multiplier by which it was produced.*

IV. *Add the partial products thus placed, and their sum will be the total product.*

(153.) Although the above rule will serve for the solution of every possible question which can occur in multiplication, yet, in particular cases, other methods may be applied, by which the process may occasionally be abridged, and which, as they are illustrations of pro-

perties of number which are otherwise useful, it may not be improper to notice here.

(154.) It sometimes happens that the multiplier is obviously the product of two or more smaller numbers, in which case the operation may be performed by successive multiplications without addition. Thus, if the multiplier be 72, we may obtain the product by first multiplying by 9 and then by 8 (138.); or, since 9 is the product of 3 and 3, we may obtain the product by multiplying successively by 3, 3, and 8. Again, since the product of 8 is 2 and 4, we may obtain the product by multiplying successively by 3, 3, 2, and 4. Let the multiplicand, for example, be 86: if we multiply by 72, by the general rule, the process is as follows: —

$$\begin{array}{r}
 \text{Multiply} \quad . \quad . \quad 86 \\
 \text{by} \quad . \quad . \quad 72 \\
 \hline
 \phantom{\text{Multiply}} \phantom{\text{by}} 172 \\
 \phantom{\text{Multiply}} \phantom{\text{by}} 602 \\
 \hline
 \text{Product} \quad . \quad . \quad \underline{6192}
 \end{array}$$

If we multiply by 9 and 8 successively, we shall have —

$$\begin{array}{r}
 \text{Multiply} \quad . \quad . \quad 86 \\
 \text{by} \quad . \quad . \quad 9 \\
 \hline
 \text{Multiply} \quad . \quad . \quad 774 \\
 \text{by} \quad . \quad . \quad 8 \\
 \hline
 \text{Product} \quad . \quad . \quad \underline{6192}
 \end{array}$$

If we multiply by 3, 3, and 8, we have —

$$\begin{array}{r}
 \text{Multiply} \quad . \quad . \quad 86 \\
 \text{by} \quad . \quad . \quad 3 \\
 \hline
 \text{Multiply} \quad . \quad . \quad 258 \\
 \text{by} \quad . \quad . \quad 3 \\
 \hline
 \text{Multiply} \quad . \quad . \quad 774 \\
 \text{by} \quad . \quad . \quad 8 \\
 \hline
 \text{Product} \quad . \quad . \quad \underline{6192}
 \end{array}$$

Finally, if we multiply by 3, 3, 2, and 4, we have—

Multiply	.	.	86
by	.	.	<u>3</u>
Multiply	.	.	258
by	.	.	<u>3</u>
Multiply	.	.	774
by	.	.	<u>2</u>
Multiply	.	.	1548
by	.	.	<u>4</u>
Product	.	.	<u>6192</u>

(155.) We have seen that the multiplier may always be resolved into parts, and the total product obtained by adding the partial products. Analogous to this is another method, which sometimes furnishes the means of considerable brevity in the process. We may first take a greater multiplier than that proposed; and, having obtained the product, we may subtract from it the partial product obtained by multiplying the multiplicand by that number by which the assumed multiplier exceeds the proposed one. This will be easily understood when applied to an example. Let us suppose that a number is required to be multiplied by 8: if we first multiply it by 10 and then by 2, and subtract the latter product from the former, it will be evident that the remainder will be the product which would be obtained by multiplying it by 8. This amounts to no more than stating, that if from ten times any thing we subtract twice that thing, eight times the same thing will remain, which is self-evident.

The application of this principle frequently presents great facility and brevity in the process of multiplication. For example, suppose it is required to multiply 387 by 299: we shall first multiply 387 by 300, and then subtract from the product 387. By the first process we have taken the multiplicand 300 times, which is once too much; if we subtract from it the multiplicand, we shall therefore get the true product: the process would be as follows:—

$$\begin{array}{r}
 \text{Multiply} \quad . \quad . \quad 387 \\
 \text{by} \quad . \quad . \quad 300 \\
 \hline
 116100 \\
 \text{Subtract} \quad . \quad . \quad 387 \\
 \hline
 \text{Product} \quad . \quad 115713
 \end{array}$$

Again: let it be required to multiply 49687 by 99999: by the ordinary method this would require five multiplications, and the addition of five lines of figures. If, however, we multiply the multiplicand first by 100000, which is done by annexing five ciphers to it, and subtract from the number thus found the multiplicand, the remainder will be the product sought: the process would be as follows:—

$$\begin{array}{r}
 4968700000 \\
 \text{Subtract} \quad 49687 \\
 \hline
 \text{Product} \quad . \quad 4968650313
 \end{array}$$

(156.) We have hitherto confined our attention chiefly to the products formed by the multiplication of two factors only. Products may, however, be formed by the continued multiplication of three or more factors. The operation which is expressed in the following manner:— $2 \times 3 \times 4$ is the *continued multiplication* of the factors 2, 3, and 4; and means that 2 is to be first multiplied by 3, and the product thus obtained to be then multiplied by 4. The result of such a process would be 24, which is, therefore, the *continued product* of 2, 3, and 4; which fact is expressed thus:—

$$2 \times 3 \times 4 = 24.$$

In like manner,

$$2 \times 3 \times 4 \times 5 = 120,$$

means that the continued product of 2, 3, 4, and 5, that is, the product of 2 and 3 multiplied by 4, and the result multiplied by 5, produces 120.

(157.) The name *factor* is extended to the numbers by the continued multiplication of which any other number is formed. Thus, 2, 3, and 4, are *factors* of 24. The *prime factors* of any number are those

prime numbers, by the continued multiplication of which the number in question is formed: since 24 is formed by the continued multiplication of 2, 3, and 4, and since 4 itself is formed by the multiplication of 2 and 2,

$$24 = 2 \times 2 \times 2 \times 3.$$

Thus the *prime factors* of 24 are 2 and 3; but 2 is 3 times repeated in the continued multiplication.

(158.) When a product is formed by the continued multiplication of the same factors, it is called a *power*: thus,

$$2 \times 2 = 4$$

$$2 \times 2 \times 2 = 8$$

$$2 \times 2 \times 2 \times 2 = 16, \text{ \&c.}$$

The numbers 4, 8, 16, &c. are powers of 2: 4 is called the *square* of 2, or *second power*; 8 is called the *cube* of 2, or *third power*; 16 is called the *fourth power* of 2; and all products in which 2 is repeated by continued multiplication, are in like manner called *powers* of 2, the numerical order of the power being determined by the number of times which 2 occurs as a factor in the continued product.

The powers of other numbers are determined in the same manner: thus,

$$3 \times 3 = 9$$

$$3 \times 3 \times 3 = 27$$

$$3 \times 3 \times 3 \times 3 = 81, \text{ \&c.}$$

9 is, therefore, the square of 3, 27 its cube, 81 its fourth power, &c. &c.

The *first power* of any number is, therefore, the number itself.

(159.) There are various ways by which the teacher may verify or *prove* the work of his pupil in multiplication; but the best are those by which the pupil is made unconsciously to verify his own work, while he at the same time is further exercised in the practice of the rule. This end may be attained in various ways, which will suggest themselves to the mind of every teacher. The same or different pupils may be made to solve the same question by different methods, and the coincidence

of the results will, in general, prove their correctness. Thus, the number which is given to one pupil as the multiplicand, may be given to another as the multiplier, and *vice versâ*. A method of verification may also be derived from the fact, that if one factor be doubled and the other halved, the product will remain unchanged, being as much increased by the one operation as it is diminished by the other. This method will always be applicable when one of the factors is an even number, — a circumstance for which it is always in the power of the teacher to provide. Thus, let the following question be proposed to one pupil: —

$$\begin{array}{r} \text{Multiply} \quad - \quad 34765 \\ \text{By} \quad - \quad - \quad 7564 \\ \hline \end{array}$$

At the same time propose to another pupil the following question, in which the multiplicand is *double* the former multiplicand, and the multiplier *half* the former multiplier: —

$$\begin{array}{r} \text{Multiply} \quad - \quad 69530 \\ \text{By} \quad - \quad - \quad 3782 \\ \hline \end{array}$$

The products must needs be the same. If they are found to differ, therefore, one or the other must be wrong, and the pupils should be made to revise their work.

(160.) Two questions may be proposed successively to the same pupil, or at the same time to different pupils, in which, with the same multiplicand, one multiplier may be double the other, or, with the same multiplier, one multiplicand will be double the other. In such case, one product must be double the other; and if not, the work must be wrong, and should be revised.

In the next chapter we shall have occasion to point out various methods by which the processes of multiplication and division may be used to verify each other.

(161.) The most easy and expeditious method of verifying complex questions in multiplication, is that which is commonly called the method of *casting out the nines*. It is performed as follows: —

Add the figures which occur in the multiplicand, and which are less than 9. In the progress of the addition, when the sum surpasses 9, omit the 9, and only carry on the remainder. You will then have, finally, a remainder less than 9. Do the same with the multiplier, and then multiply the two remainders together; divide this product by 9, and find the remainder. In like manner, add the figures of the product, casting out the nines — a remainder will be obtained less than 9. If this be the same as the remainder found by dividing the product of the remainders in the multiplicand and multiplier, then it may be considered, generally, that the work is correct. To make the above explanation intelligible, let us suppose that the multiplicand and multiplier are 23707 and 4567, the product being 108269869. Adding the figures of the multiplicand, we have 7 and 7 make 14; rejecting 9, the remainder is 5. This added to 3 and 2 gives 10; rejecting the 9, the remainder is 1. Proceeding in the same way with the multiplier, we have 7 and 6 make 13; rejecting the 9, we have a remainder 4, which, added to 5, makes 9; rejecting this, we have the last figure, 4, remaining. The remainders, therefore, in the multiplicand and multiplier, are 1 and 4, which multiplied together give 4: this being less than 9, has no nines to be rejected. Proceed in the same manner with the product. The first figure, 9, is neglected: 6 and 8 are 14, which is 5 above 9. The next figure, 9, is neglected, and 5 is added to 6, which gives 11. Carrying the 2 above 9 to the next figure, we obtain 4, which added to 8 gives 12. Carrying the 3 above 9 to the 1, we obtain 4, which is the same remainder as that obtained from the multiplier and multiplicand.*

* The truth of this method may be established in the following manner: —

1. If a number be divided by 9, the same remainder will be obtained as if the sum of its digits were divided by 9.

Suppose the number is 2376. It may be resolved into the following parts: —

$$\begin{aligned} &2 \times (999 + 1) \\ &3 \times (99 + 1) \\ &7 \times (9 + 1) \\ &6 \end{aligned}$$

If the remainder which is obtained after casting out the nines from the product, be *not* the same as the remainder obtained as above described from the factors, then it is certain that the work must be incorrect ; but, on the other hand, if the remainder be the same in both cases, still it is not absolutely certain, though in the highest degree probable, that the work is correct. It may happen by possibility that two errors may occur in the product which will compensate each other, so far as they produce any effect on the remainder after casting out the nines. Thus, if one figure of the product be less than it ought to be by 1, while another figure is greater than it ought to be by 1, then the sum of the digits will remain the same as if the product were correct ; and therefore the remainder, after casting out the nines, will not be affected. It is true that such a coincidence of errors as would produce this compensation is highly improbable, and therefore the method may be used in teaching as a sufficiently certain means of verification ; but when calculations are made for

or, what is the same,

$$2 \times 999 + 3 \times 99 + 7 \times 9 + 2 + 3 + 7 + 6.$$

Now it is evident, that 2×999 , 3×99 , and 7×9 , are severally multiples of 9 ; and, consequently, when divided by 9 would leave no remainder. When the entire number 2376 is divided by 9, the remainder must therefore be the same as it would be if $2 + 3 + 7 + 6$ were divided by 9.

2. If the two factors be considered as consisting of multiples of 9 and remainders, the product will consist of a multiple of 9, and the product of the same remainders.

Let the factors be 357 and 254. The greatest multiple of 9 contained in the former is 351, and in the latter 252. The numbers to be multiplied are then $351 + 6$ and $252 + 2$. We must multiply the former, first by 252, which gives the products $351 \times 252 + 6 \times 252$, and next by 2, which gives the products $351 \times 2 + 6 \times 2$. The total product is then as follows : —

$$351 \times 252 + 6 \times 252 + 351 \times 2 + 6 \times 2.$$

Now, since 351 and 252 are each of them multiples of 9, it follows that 351×252 , 6×252 , and 351×2 , are severally multiples of 9. The sum of these, therefore, is a multiple of 9 ; and therefore the whole product consists of a multiple of 9, and the product of the remainders (6 and 2) found by dividing the factors by 9.

3. If the factors be divided by 9 respectively, and the remainders be multiplied together and divided by 9, the same remainder will be obtained as if the product of the factors were divided by 9.

It has been just proved that the product of the factors consists of a multiple of 9 and the product of the remainders. The former divided by 9 has no remainder ; therefore, if the whole product be divided by 9, the same remainder will be found as if the product of the two remainders were divided by 9.

4. But the same remainders will be found if the sums of the digits of the factors and the product be divided by 9, as if these numbers themselves were divided by 9.

(164.)—TABLE OF SQUARES AND CUBES.

No.	Square.	Cubes.	No.	Square.	Cubes.	No.	Square.	Cubes.
2	4	8	35	1225	42875	68	4624	314432
3	9	27	36	1296	46656	69	4761	328509
4	16	64	37	1369	50653	70	4900	343000
5	25	125	38	1444	54872	71	5041	357911
6	36	216	39	1521	59319	72	5184	373248
7	49	343	40	1600	64000	73	5329	389017
8	64	512	41	1681	68921	74	5476	405224
9	81	729	42	1764	74088	75	5625	421875
10	100	1000	43	1849	79507	76	5776	438976
11	121	1331	44	1936	85184	77	5929	456533
12	144	1728	45	2025	91125	78	6084	474552
13	169	2197	46	2116	97396	79	6241	493049
14	196	2744	47	2209	103823	80	6400	512000
15	225	3375	48	2304	110592	81	6561	531441
16	256	4096	49	2401	117649	82	6724	551368
17	289	4913	50	2500	125000	83	6889	571787
18	324	5832	51	2601	132651	84	7056	592704
19	361	6859	52	2704	140608	85	7225	614125
20	400	8000	53	2809	148877	86	7396	636056
21	441	9261	54	2916	157464	87	7569	658503
22	484	10648	55	3025	166375	88	7744	681472
23	529	12167	56	3136	175616	89	7921	704969
24	576	13824	57	3249	185193	90	8100	729000
25	625	15625	58	3364	195112	91	8281	753571
26	676	17576	59	3481	205379	92	8464	778688
27	729	19686	60	3600	216000	93	8649	804357
28	784	21952	61	3721	226981	94	8836	830584
29	841	24389	62	3844	238328	95	9025	857375
30	900	27000	63	3969	250047	96	9216	884736
31	961	29791	64	4096	262144	97	9409	912673
32	1024	32768	65	4225	274625	98	9604	941192
33	1089	35937	66	4356	287496	99	9801	970299
34	1156	39304	67	4489	300763	100	10000	1000000

CHAP. VI.

DIVISION.

(165.) IN division there are two numbers given ; one of which is called the *dividend*, or *number to be divided*, and the other the *divisor*, or *number by which the dividend is to be divided*. The result of the operation, or number sought, is called the *quotient*, or *quote*. This arithmetical operation may be considered under several different points of view.

(166.) The dividend being regarded as the product of two factors, one of which is the divisor, the object of division is to discover the other factor ; when found, the other is the quotient. Thus, if 48 be the dividend, and 8 the divisor, then that number which, being multiplied by 8, gives 48 as the product, is the quotient ; — that number is 6.

(167.) Hence it follows, that by whatever means the quotient is found, the process may be verified by multiplying the divisor by the quotient. The product should be the dividend ; and if it be not, the operation must have been incorrectly performed.

(168.) Since the product of the divisor and quotient is the dividend, it follows that the quotient, repeated as often as there are units in the divisor, will make up the dividend. Hence the process of division is presented under another point of view ; being that process by which the dividend is divided into as many equal parts as there are units in the divisor, one of those equal parts being the quotient. It is from this mode of considering the operation that it has received the name *division*.

(169.) It likewise follows, that the divisor being repeated as often as there are units in the quotient, a number will be obtained equal to the dividend : hence

the process is exhibited under another aspect, being the operation by which we may discover how often the divisor must be repeated in order to make up the dividend, or, as it is commonly expressed, how often the divisor is *contained in* the dividend.

Under this point of view, division may be considered as continued subtraction, in the same manner as multiplication is continued addition. If we wish to find how often 8 is contained in 48, we shall subtract 8 from 48, and then subtract 8 from the remainder, and so continually subtract 8 from every remainder until the subtrahend is exhausted. The number of times which the divisor 8 has then been subtracted from the dividend is the same as the number of times which the divisor is *contained in* the dividend.

(170.) But from these ways of considering the process of division, it would appear that the operation could never be performed, except in the particular case in which the dividend happens to be the product of two numbers, one of which is the divisor. Now, in by far the greater number of cases which can be proposed, this circumstance will not happen. If the dividend be 50, and the divisor 8; there is no number (so far as the definitions of number already given extend) which, being multiplied by the divisor, would produce the dividend; and therefore the division, in the sense in which that operation has been just defined, would be impossible. We shall see hereafter, however, that by enlarging our ideas of number, we shall be enabled to perform division even in this case: meanwhile it is usual to effect a partial division in those cases in which the dividend is not an exact multiple (121.) of the divisor. The quotient which is obtained in such cases is a partial quotient, expressing merely the number of times which the divisor is contained in the dividend (129.). In such cases there will always be a remainder, being that part of the dividend which would remain if the divisor were subtracted as often as possible from it.

Let us suppose, for example, that the dividend is 50:

by successively subtracting 8 from the dividend, we should find that after subtracting it 6 times, there would be a remainder 2 : from this, 8, being a greater number, could not be subtracted. When it is proposed, then, to divide 50 by 8, we say that the quotient is 6, and the remainder 2 ; meaning by the word quotient, not the complete quotient of 50 divided by 8, but the exact quotient of 48 divided by 8, 48 being the greatest multiple of 8 contained in 50. The incompleteness of the division is always signified by declaring what is the remainder at the same time that the quotient is expressed. Although, in such cases, the quotient is, strictly speaking, a partial quotient, yet it is customary to call it simply the quotient, since no mistake can arise, provided that the remainder at the same time be declared.

(171.) From what has been here explained, it will follow, that when the dividend is not an exact multiple of the divisor, the product of the quotient and divisor is the greatest multiple of the divisor contained in the dividend : it is also the greatest multiple of the quotient contained in the dividend. It also follows, that when the product of the quotient and divisor is subtracted from the dividend, the remainder will be the same as the remainder obtained in the ordinary process of division ; and that, if to the product of the quotient and divisor the remainder be added, the sum will be the product.

Let the dividend be 68, and the divisor 9 ; the quotient will be in this case 7, and the remainder 5. Now 9×7 is the greatest multiple of 9 or of 7 contained in 68. If this product be subtracted from 68, the remainder will be 5 ; for if 5 be added to this product, the sum will be the dividend 68.

(172.) Hence, to verify the process of division, multiply the divisor by the quotient, and to this product add the remainder : the sum should be the dividend ; and if it be not, there must be some error in the work.

(173.) The work may also be verified in the following manner : — Subtract the remainder from the divi-

dend ; the number remaining should be the exact product of the divisor and quotient. If this number be divided by the quotient, the result should be the divisor ; or if it be divided by the divisor, the result should be the quotient ; or, finally, if the divisor and the quotient be multiplied together, this same number should be the result. Any of these methods may be used for verification.

Having stated and proved the general principles on which the process of division rests, we shall now proceed to explain the practical methods by which, when the divisor and dividend are given, the quotient may be discovered.

(174.) If the divisor be a single digit, and the dividend do not exceed 100, then the quotient and remainder may generally be found by the knowledge of the multiplication table carried as far as 10 times 10.

If the dividend in this case happen to be an exact multiple of the divisor, then it will be remembered as one of the products in the multiplication table, and it will be recollected what that number is, by which the divisor being multiplied, the product is the dividend. Thus, suppose the dividend is 72, and the divisor 8 ; the question is, what number multiplied by 8 produces 72 ? and the knowledge of the multiplication table immediately suggests that this number is 9 : the quotient, therefore, is in this case 9, there being no remainder.

The discovery of the quotient is not more difficult when the dividend is not an exact multiple of the divisor. Let us suppose that the dividend is 60, and the divisor 8 ; the question then is, what is the greatest number which, being multiplied by 8, will give a product less than 60 ? Now, we know from the multiplication table, that 7 multiplied by 8 gives the product 56, and that 8 multiplied by 8 gives the product 64 : consequently 56 is the greatest multiple of 8 contained in 60. If 60, therefore, be divided by 8, the quotient will be 7, and the remainder 4.

(175.) We have seen, that if the dividend be an ex-

act multiple of the divisor, the division will be completed without a remainder, and the quotient will express the number of times which the divisor is contained in the dividend. If in this case the dividend be multiplied by 10, the quotient must also be multiplied by 10, since the divisor will be contained 10 times as often in a dividend 10 times as great. In the same manner, if the dividend be multiplied by 100, the quotient must also be multiplied by 100, for a like reason, and so on. Thus, it appears that in cases where there is no remainder, the division will remain correct, if we annex the same number of ciphers to the dividend and quotient (65.). Thus, if the divisor be 2 and the dividend 8, the quotient will be 4, since 2 is contained 4 times in 8 : if we annex a 0 to the dividend, we multiply it by 10, and 2 is contained in 80 ten times as often as it is contained in 8 ; therefore it is contained 40 times in 80, and the quotient will be 40, being the former quotient with a cipher annexed. In like manner, if we annex 2 ciphers to the dividend, we multiply it by 100, and the divisor 2 will be contained in 800 a hundred times as often as it is contained in 8, and the quotient will accordingly be 400, which is the first quotient with 2 ciphers annexed. We shall presently perceive the use of these inferences.

(176.) When the dividend is a large number, it is necessary in the practical process of division to resolve it, or to imagine it resolved, into several smaller ones, and the division is in that case effected by dividing each of the smaller numbers separately by the divisor, and then adding together the quotients. Thus we might divide 18 by 3, by resolving it into 3, 6, and 9, which added together would make up 18. 3 is contained in 3 once, in 6 twice, and in 9 three times : the three quotients are, therefore, 1, 2, and 3 ; and these added together make 6, which is the total quotient of 18 divided by 3.

(177.) Let us now consider the case in which the divisor is a single digit, and the dividend a number

consisting of several digits. Let the divisor, for example, be 3, and the dividend 963 : we shall conceive this dividend resolved into 3 parts, — 900, 60, and 3. In writing down the process of division, it is usual to place the divisor on the left of the dividend, separated by a line, and to write the quotient either under the dividend, or separated by a line to the right of the dividend. Taking the three partial dividends, into which we have resolved the total dividend 963, the three partial divisions would be written as follows ;—

$$3 \) \ 900 \ (\ 300 \qquad 3 \) \ 60 \ (\ 20 \qquad 3 \) \ 3 \ (\ 1$$

We know that 3 is contained 3 times in 9, without a remainder ; and, therefore, by what has been proved (175.), it is contained 300 times in 900. For the same reason, since 3 is contained twice in 6, without a remainder, it is contained 20 times in 60. It is evident, then, that the three partial quotients are 300, 20, and 1, and the total quotient 321.

In order to make the principle on which the process rests intelligible, we have here written it at greater length than is necessary in practice : all the three partial divisions may be written and performed as follows :—

$$\begin{array}{r} 3 \) \ 963 \\ \underline{321} \end{array}$$

The figure 9 of the dividend here expresses 900, and 3 being contained in it 300 times without a remainder, we place 3 in the hundreds' place of the quotient ; we do not annex two ciphers to it, because the places occupied by those ciphers are to be filled up by the other two partial quotients. In the same manner 6 in the dividend signifies 60, which divided by 3 gives the quotient 20 : we therefore place 2 in the tens' place of the quotient, leaving the units' place still unoccupied, to be filled by the third partial quotient, which in this case being 1, the division is completed as above.

(178.) It might so happen that the third partial dividend might be wanting, and that the dividend would only be

resolved into the first two partial dividends: this would happen if the last figure of the dividend were 0. In that case, the two partial dividends would be 900 and 60, and the two partial quotients would be 300 and 20; these would be expressed by placing 3 and 2 in the hundreds' and tens' place of the quotient; but as no third partial quotient would in that case be furnished to fill the units' place of the total quotient, it would be necessary to fill that place by 0, otherwise the 3 and 2 would express 30 and 2, and not 300 and 20.

(179.) Again: it might so happen, that the second partial dividend would be wanting, the first and third remaining: this would happen if the dividend had been 903; in that case it would be resolved into the partial dividends 900 and 3, and the two partial quotients would be 300 and 1. In that case it would be necessary to write the 3 in the hundreds' place, and the 1 in the units'; but it would be obviously necessary to fill the tens' place by a 0.

(180.) In the examples which we have here given, it will be observed that each of the digits of the dividend is an exact multiple of the divisor. This circumstance gives a simplicity to the example, which renders it proper as a first step in the explanation of the practical process. Let us now, however, take the more general case, in which the digits of the dividend are not exact multiples of the divisor.

Let the dividend be 762, and let the divisor, as before, be 3: following the steps taken in the former example, we should first naturally resolve this dividend into three parts:—

700 60 2

But 700 not being an exact multiple of the divisor, we adopt another expedient: the number next less than 7 which is divisible by 3 is 6; 3 is contained in 6 twice, and 200 times in 600: instead, therefore, of taking 700 as the first partial dividend, we shall take 600, and add the remaining 100 to the second partial dividend; the

three parts into which the original dividend is resolved will then be

600 160 2

The second of these, 160, is, however, not an exact multiple of 3; the greatest multiple of 3 contained in 16 is 15; and since 3 is contained 5 times in 15, it will be contained 50 times in 150: we shall therefore take 150 as the second partial dividend, and transfer the remaining 10 to the third part 2, by which it becomes 12, and the original dividend will then be resolved into the following parts:—

600 150 12

these being severally divided by 3, give the following quotients:—

200 50 4

The total quotient will be obtained by adding together these three partial quotients: it is, therefore, 254.

But in practice this process may be written in a very abridged form. To obtain the first quotient, 200, it is only necessary to divide 7 by 3, and put the quotient 2 in the hundreds' place, reserving the remainder 1. Conceive this remainder prefixed to the digit 6, which fills the tens' place, and divide the number 16 thus found by 3; place the quotient 5 in the tens' place of the quotient, and prefix the remainder 1 to the figure 2, which fills the units' place of the dividend: divide 12, the number thus obtained, by 3, and put the quotient 4 in the units' place of the quotient. It must be evident that in this way the same process is performed as that which was more fully developed above.

In general, then, when the dividend consists of several digits, the divisor being a single digit, proceed by the following

RULE.

(181.) *Divide the first figure of the dividend by the divisor, and place the quotient under the same figure of the dividend; prefix the remainder to the next figure of the dividend, and divide the number thus obtained by the divisor; place the quotient under the second figure of*

the dividend, and prefix the remainder to the third figure of the dividend ; divide the number thus obtained by the divisor, and proceed as before, continuing this process until you arrive at the units' place of the dividend, when the division will be complete.

(182.) When it happens that the first figure of the dividend is less than the divisor, the first figure of the quotient would be 0, since the divisor is not contained in a number less than itself ; but as 0 standing on the left of a number does not affect its value, it is unnecessary to insert it. The following example will illustrate this : — Let the dividend be 162, the divisor being 3 ; by following the above rule strictly, the process of division would be as follows : —

$$\begin{array}{r} 3 \) \ 162 \\ \underline{054} \end{array}$$

In performing the division, we should say, the quotient of 1 by 3 is 0, with a remainder 1, which being prefixed to 6, we divide 16 by 3, &c. But since the 0 prefixed to 54 has no signification, it is omitted, and we begin the process of division, not by expressing the division of 1 by 3, but of 16 by 3. Whenever, then, the first figure of the dividend is less than the divisor, the first step of the process should be to divide the first two figures of the dividend by the divisor, and in that case the quotient should be placed under the second figure of the dividend. The above example would, therefore, be written thus : —

$$\begin{array}{r} 3 \) \ 162 \\ \underline{54} \end{array}$$

(183.) The rule above given will still be applicable if any of the places of the dividend happen to be filled by ciphers. Take the following example : —

$$\begin{array}{r} 5 \) \ 980700 \\ \underline{196140} \end{array}$$

The first two figures of the quotient are found by the rule already explained. When 9 is divided by 5 there is a remainder 4, which, prefixed to 8, gives 48 ; this di-

vided by 5 gives 9, with a remainder 3; this remainder must be prefixed to the following 0: the 30 thus obtained, divided by 5, gives 6 for the next figure of the quotient: the 2 which remains when 7 is divided by 5 must be prefixed to the following 0, which gives 20; this divided by 5 gives the quotient 4, without a remainder. But as there is another final 0 in the dividend, a 0 must be also annexed to the quotient. (178.)

(184.) Let us now consider the most complex case of division, in which the divisor and dividend are both numbers which consist of several places. As in the former cases, the division is here effected by resolving the dividend into a number of parts, each of which is an exact multiple of the divisor. We shall first explain the method of thus resolving the dividend into parts, which will be easily understood by its application to an example. Let the dividend be 4967398, and the divisor 37: it is required to resolve the former into several numbers each of which will be an exact multiple of 37. To effect this, let it be considered that if any number be a multiple of another, the same number will still be a multiple of the other when any number of ciphers are annexed to it: thus, if 8 be a multiple of 4, 80, 800, 8000, &c. will also be multiples of 4 (175.). This is sufficiently evident.

The practical process by which the dividend is resolved into multiples of the divisor may be written as follows: —

From	-	-	A	-	4967398
Subtract		•	B	-	3700000
From	-	-	C	-	1267398
Subtract		-	D	-	1110000
From	-	-	E	-	157398
Subtract		-	F	-	148000
From	-	-	G	-	9398
Subtract		-	H	-	7400
From	-	-	I	-	1998
Subtract		-	K	-	1850
			L	-	148

The number *A* is the dividend itself: we can find one multiple of the divisor which is contained in it by finding the greatest multiple of the divisor contained in 49, and annexing to it as many noughts as there are remaining places in the dividend. Now, 37 is contained only once in 49; if, therefore, to 37 we annex five ciphers, we shall obtain a multiple of the divisor (175.) less than the dividend *A*, and which is therefore contained in it: this, then, is one of the parts into which we wish to resolve the dividend. We accordingly subtract this first multiple, *B*, of the divisor from *A*, and we get the remainder *C*; this number *C* is now what remains of the dividend to be resolved into multiples of the divisor. We shall obtain another multiple of the divisor contained in *C* if we can ascertain how often the divisor is contained in the initial figures of *C*: the first two figures of *C* forming a number less than the divisor, we must take the first three figures; and the question is, how often 37 is contained in 126: this can only be ascertained by trial; but we may form a near guess at it by enquiring how often 3 is contained in 12. Since 12 divided by 3 gives the quotient 4, we must try whether 4 times 37 are less than 126: we find, however, that they are greater, and therefore infer that 37 is contained less than 4 times in 126. We next try if it be contained 3 times: we find that 3 times 37 are 111, which, being less than 126, is the greatest multiple of 37 contained in that number. Now, since 111 is a multiple of 37, 1110000 is also a multiple of 37, and this number is less than the first remainder *C*: it is, therefore, a multiple of 37 contained in *C*. We have thus found two multiples of 37, *B* and *D*, which form parts of the dividend *A*: by subtracting *D* from *C* we shall obtain what remains of the dividend to be resolved into multiples of 37: this remainder is *E*, and we proceed with it as in the former cases, ascertaining how often 37 is contained in 157. Observing that 3 is contained 5 times in 15, we try whether 37 is contained 5 times in 157; but we find that 5 times 37 are greater than 157: we next try whether 37 is contained

4 times in 157; and finding that 4 times 37 are 148, we infer that 148 is the greatest multiple of 37 contained in 157. Annexing to 148 as many ciphers as there are remaining places in E, we obtain a multiple of 37 which is contained in E: we have thus obtained 3 multiples of 37, B, D, and F, which form parts of the original dividend, and we shall obtain what remains of that dividend to be resolved into parts by subtracting F from E; the remainder is the number G; we proceed with it in the same manner, and enquire how often 37 is contained in 93: since 3 is contained 3 times in 9, we enquire if 37 be contained 3 times in 93; but finding that 3 times 37 are greater than 93, we take twice 37 or 74, which, being less than 93, is contained in it. Annexing as many ciphers to this as there are remaining places in the number G, we obtain the number H, which is a multiple of 37. We have thus obtained four multiples of 37, B, D, F, and H, which are parts of the original dividend: the remaining part of the dividend not yet resolved into multiples of the divisor is I, found by subtracting H from G. We proceed with this as in the former case, enquiring how often 37 is contained in 199: we find that it is contained 5 times; and 185 being a multiple of 37, the same number with a cipher annexed is also a multiple of 37. We thus obtain K, another multiple of 37, which is a part of the original dividend: the remainder L is itself 4 times 37, and is therefore a multiple of the divisor. We have thus resolved the original dividend into the parts B, D, F, H, K, L, which are respectively multiples of the divisor, and which when added together, must reproduce the original dividend: —

B	-	-	-	-	3700000
D	-	-	-	-	1110000
F	-	-	-	-	148000
H	-	-	-	-	7400
K	-	-	-	-	1850
L	-	-	-	-	148
A	-	-	-	-	<u>4967398</u>

K

In order to find how often the divisor is contained in the original dividend A, it is now only necessary to find how often it is contained in the several parts B, D, F, H, K, and L of that dividend; in other words, taking these numbers severally, as partial dividends, and dividing them by 37, we shall obtain as many partial quotients; which being added together, will give the total quotient of the dividend A, when divided by the divisor 37. The process at length would be as follows: —

Divisor.	Dividend.	Quotient.
37	3700000	100000
37	1110000	30000
37	148000	4000
37	7400	200
37	1850	50
37	148	4
37) 4967398 (134254 .	

With a view to explain in the clearest manner the principles on which this process rests, we have here expressed it at much greater length than is necessary in its practice after those principles have been understood. It will be observed, that each of the partial divisions furnishes the successive figures of the total quotient followed by ciphers. Thus the last partial quotient would be expressed by placing 4 in the units' place of the first, the preceding one by placing 5 in the tens' place, and so on. In practice, therefore, instead of writing the several partial quotients separately, we omit the ciphers in the first, and in their places put the first figures of all the others. It is likewise obviously unnecessary to write the divisor before each of the partial dividends, as its presence may be understood after being once written before the first. The process of resolving the dividend into parts which are severally multiples of the divisor, and the actual process of the several partial divisions, may also be combined in the same written arrangement. Observing these abridgments, the practical process of division would take the following form: —

$$\begin{array}{r}
 37 \) \ 4967398 \ (\ 134254 \\
 \underline{3700000} \\
 1267398 \\
 \underline{1110000} \\
 157398 \\
 \underline{148000} \\
 9398 \\
 \underline{7400} \\
 1998 \\
 \underline{1850} \\
 148 \\
 \underline{148} \\
 \dots
 \end{array}$$

It will be further observed, that in each successive remainder the latter figures remain unchanged throughout the process, the initial figures only being affected by the subtractions. It is, therefore, unnecessary to subtract in any case more than three figures of each remainder each time, provided that the next figure of the dividend is always annexed to the remainder. The process according to this abridgment would be thus expressed:—

$$\begin{array}{r}
 37 \) \ 4967398 \ (\ 134254 \\
 \underline{37} \\
 126 \\
 \underline{111} \\
 157 \\
 \underline{148} \\
 93 \\
 \underline{74} \\
 199 \\
 \bullet 185 \\
 \underline{148} \\
 148 \\
 \dots
 \end{array}$$

By placing 37 under 49, the former is here made to express 3700000, which is the same number as it expressed when the process was written down in full. We subtract 37 from 49, and to the remainder, 12, we annex the next figure, 6, of the dividend: by placing the second

dividend, 111, under 126, or under 496 of the original dividend, it is made to express 1110000. In the same manner the third dividend, 148, being placed under 967 of the original dividend, expresses 148000; and so on.

To fix the ideas of this arithmetical process clearly and firmly in the mind of the pupil, we shall now take another example, in which the divisor shall consist of 3 figures : —

$$\begin{array}{r}
 256 \) \ 890368 \ (\ 3478 \\
 \underline{768} \\
 1223 \\
 \underline{1024} \\
 1996 \\
 \underline{1792} \\
 2048 \\
 \underline{2048} \\
 0
 \end{array}$$

We begin by taking the first 3 figures of the dividend, and, considering them as one number, we enquire how often the divisor is contained in them : this can only be determined, as before, by trial ; but we shall in some measure be led to the knowledge of the number sought by trying how often the first figure of the divisor is contained in that of the dividend. Since 2 is contained 4 times in 8, we shall first try whether 256 is contained 4 times in 890 ; but by multiplying 256 by 4 we obtain a product greater than 890 : therefore 256 is not contained 4 times in 890. We next try whether it is contained 3 times in 890 ; and by multiplying 256 by 3 we obtain the product 768, which, being less than 890, must be the greatest multiple of 256 contained in that number : we place 3 as the first figure of the quotient, and subtract 768 from 890 ; but, in doing this, it is evident that we have, in fact, subtracted 768000 from 890000, such being the local values of the digits engaged in the operation. The 3 which is placed in the quotient should, therefore, signify 3000, and we should accordingly write three ciphers after it ; but it is unnecessary to write these, because their places will be filled

by the other figures of the quotient, which will be discovered in the course of the operation. To the remainder 122 we annex the succeeding figure of the dividend, so that the second partial dividend will be 1223. We must now try how often 256 is contained in this number; and since 2 is contained in 12 six times, we should try whether 256 be contained in 1223 six times: we find, however, that by multiplying 256 by 6, or even by 5, we obtain a product greater than 1223. If, however, we multiply 256 by 4, we obtain a product less than 1223: 4 is, therefore, the next figure of the quotient; and we obtain the corresponding partial dividend, 1024, by multiplying the divisor by 4. By continuing the process in the same manner, we obtain the total quotient as above.

In the examples which we have given above, it happens that every figure of the quotient is a significant digit. Although the same principles will be applicable in the cases in which ciphers occur, yet the student might be embarrassed if he were not practically prepared for that circumstance. The following example will illustrate the mode of proceeding in such a case:—

$$\begin{array}{r}
 2465 \) \ 123262325 \ (\ 50005 \\
 \underline{12325} \\
 12325 \\
 \underline{12325} \\
 12325
 \end{array}$$

The first 4 figures of the dividend forming in this case a less number than the divisor, we must try how often the divisor is contained in the number formed by the first 5 figures. Since 2 is contained in 12 six times, we shall try if the divisor is contained in that number 6 times: we find, however, that it is not, but that it is contained in it 5 times. Multiplying the divisor, then, by 5, the first figure of the quotient, we write the product under the first 5 figures of the dividend; and performing the subtraction, we obtain the remainder 1. To this remainder, according to the process already ex-

plained, we should annex the succeeding figure of the dividend ; but this gives 12, a number much less than the divisor. Under these circumstances, we must continue to annex to the remainder the successive figures of the dividend, until we obtain a number which is not less than the divisor. To accomplish this in the present instance, it is necessary to annex all the remaining figures of the dividend ; and a number is thus obtained which is exactly 5 times the divisor. This number is, then, the second partial dividend, and the corresponding quotient is 5, which, as it should express units, ought to stand in the units' place of the quotient. The first quotient, however, should express 50,000, and should therefore stand in the ten thousands' place. Now, in order to express the proper values of these two digits, it is necessary to place between them 3 ciphers ; and the quotient is accordingly the number above written. In general, in such cases, the method of proceeding is as follows : — When, on annexing a figure from the dividend to any remainder, you find that that remainder is less than the divisor, you must immediately annex a cipher to the quotient, and annex the next figure of the dividend to the remainder. If the remainder still continue to be less than the divisor, you must annex another 0 to the quotient ; and you must continue to do this until, by successively annexing figures of the dividend to the remainder, you obtain a number greater than the divisor.

The following example will illustrate these observations : —

$$\begin{array}{r}
 6437 \) \ 19343313785059 \ (\ 3005020007 \\
 \underline{19311 \dots\dots\dots} \\
 32313 \\
 \underline{32185} \\
 12878 \\
 \underline{12874} \\
 45059 \\
 \underline{45059} \\
 \dots\dots
 \end{array}$$

Proceeding in the usual way, the first remainder is 32. Annexing to this the next figure, 3, of the dividend, the remainder is still less than the divisor; we must therefore annex a 0 to the quotient, and *bring down*, as it is called, the next figure, 1, of the dividend. It will be found convenient in working questions of this kind, and, in general, in long division, to mark with a dot each figure of the dividend which is successively annexed to the remainders: by this means no mistake can be made in selecting the figures to be annexed.

In order to render the first remainder in the present example greater than the divisor, it is necessary to bring down 3 figures from the dividend. The first 2 of these make it necessary to add noughts to the quotient, and, when the third is annexed, the divisor is contained in the number so formed. 5 is then the next figure of the quotient: the next remainder requires 2 figures to be brought down to render it greater than the divisor; one 0 is therefore added to the quotient, and the divisor being contained in the number thus formed twice, the succeeding figure of the quotient is 2. The next remainder is 4, and the 4 remaining figures of the dividend must be brought down in order to get a number greater than the divisor: 3 noughts are therefore annexed to the quotient, and the divisor is found to be contained exactly 7 times in the number thus formed: 7 is therefore the last figure of the quotient.

(185.) In certain particular cases the process of division may be greatly facilitated by the peculiar nature of the divisor or dividend. We have already seen that a number may be multiplied by 10, 100, 1000, &c. by annexing to it a corresponding number of ciphers. It follows, therefore, that a number terminating in ciphers may be divided by 10, 100, 1000, &c. by expunging a corresponding number of ciphers: that is, it may be divided by 10 by cutting off 1 cipher, by 100 by cutting off 2 ciphers; and so on.

Whether a number terminates in ciphers or not, however, the division by 10, 100, 1000, &c. is not more

difficult. Let us suppose that we wish to divide 3567 by 10. We first subtract 7 from it, and it becomes 3560 : the latter is divided by 10 by omitting the 0 : therefore, if 3567 be divided by 10, the quotient will be 356, with a remainder 7. In the same manner, if we would divide it by 100, we shall suppose 67 in the first instance subtracted from it. The remainder, 3500, will be divided by 100, by omitting the 2 ciphers : thus, if 3567 be divided by 100, the quotient will be 35, with a remainder 67. In general, then, it will be easily perceived, that to divide any number by 10, 100, 1000, &c. we must cut off as many figures on the right as there are ciphers in the divisor. The figures thus cut off will be the remainder in the division, and the other figures will be the quotient.

(186.) As multiplication may be performed by multiplying successively by the factors of the multiplier (138.), so division may be also performed by dividing successively by the factors of the divisor. By attending to this circumstance, the process of division may very often be considerably abridged. Thus, if we wish to divide by such a number as 72, we may divide first by 9, and then divide the quotient thus obtained by 8. If this process be compared with the general method already explained, it will be perceived that it is much more expeditious and concise. Let the dividend be 62000478, and the divisor 81. Instead of dividing directly by 81, we shall divide the dividend by 9, and the quotient thus obtained also by 9. The process will be as follows : —

$$\begin{array}{r} 9 \) \ 62000478 \\ 9 \) \ \underline{6888942} \\ \quad 765438 \end{array}$$

In the example just given there is no remainder. In applying this method to cases in which there is a remainder, the process is still very simple and brief. Let it be required to divide 3763 by 72 : we shall first divide it by 9, and then divide the quotient by 8 ; the process will be as follows : —

$$\begin{array}{r}
 9 \) \ 3763 \\
 8 \) \ 418 \quad - \quad 1 \text{ remainder} \\
 \underline{\quad 52 \quad} \quad - \quad 2 \text{ ditto}
 \end{array}$$

The quotient is 52 ; but neither of the above remainders would be the actual remainder in the division by 72. The actual remainder, however, may be easily found by the following rule : — *Multiply the second remainder above by the first divisor, and to the product add the first remainder ; the sum will be the remainder in the actual division ;* which, in the present case, would be 19.*

(187.) The operations of multiplication and division may frequently be brought in aid of each other, so that, by combining both, the result may be obtained more concisely than it could be by either separately. For example, if we wish to multiply a number by 25, we may proceed thus : — First annex 2 ciphers to it, and then divide by 4. Let the number to be multiplied be 56738. According to the ordinary method of multiplication, the process would be as follows : —

$$\begin{array}{r}
 56738 \\
 25 \\
 \hline
 283690 \\
 113476 \\
 \hline
 1418450
 \end{array}$$

According to the method above explained, the process would take the following more abridged form : —

$$\begin{array}{r}
 4 \) \ 5673800 \\
 \underline{\quad 1418450 \quad}
 \end{array}$$

* The proof of this will be easily understood by those who have some knowledge of algebraical notation. Let the dividend be D , the divisor d , the quotient q , and the remainder r . Let the two factors of d be d' and d'' , so that $d = d' d''$. When D is divided by d' , let the quotient be q' , and the remainder r' . When q' is divided by d'' , the quotient will be q , and let the remainder be r'' . By the nature of division we have the following relations : —

$$\begin{array}{ll}
 & d' q' + r' = D, \\
 & d'' q' + r'' = q'; \\
 \text{therefore} & - \quad d' d'' q' + d' r'' + r' = D; \\
 \text{but since} & - \quad d' d'' = d, \\
 \text{we have} & - \quad d q + d' r'' + r' = D; \\
 \text{also} & - \quad d q + r = D; \\
 \text{therefore} & - \quad d' r'' + r' = r;
 \end{array}$$

which is the rule expressed in the text.

The principle on which this method rests is easily understood. We are required to multiply by 25 ; but in multiplying by 100 in the first instance, which we do by annexing 2 ciphers, we have obtained a product four times too great, since 100 is 4 times 25 : wherefore it is necessary to divide the product thus obtained by 4, in order to get the true product.

Again, suppose it is required to multiply 67389 by 125, the ordinary process would be as follows : —

$$\begin{array}{r}
 67389 \\
 125 \\
 \hline
 336945 \\
 134778 \\
 67389 \\
 \hline
 8423625
 \end{array}$$

This question may, however, be more briefly solved in the following manner : — Annex 3 ciphers to the multiplicand, and divide by 8. This process will then be —

$$\begin{array}{r}
 8 \) \ 67389000 \\
 \hline
 8423625
 \end{array}$$

In this case, by annexing 3 ciphers, we multiply the multiplicand by 1000, which is 8 times the proposed multiplier ; consequently the product thus obtained, being divided by 8, gives us the true product.

(188.) In similar cases of division like methods may be adopted for abridging the process. If we wish to divide a number by 25, we may first divide it by 100, and then multiply the quotient by 4 : for, by dividing by 100, we divide by a number 4 times greater than that proposed, and therefore must multiply the quotient by 4, in order to obtain the true quotient. In the same manner, to divide by 125 we may divide by 1000, and multiply the result by 8 for like reasons.

If the dividend ends in 2 or more ciphers, this process is extremely short and simple. Let the dividend be 7637500, and the divisor 25 : we divide by 100, by cutting off the 2 ciphers, and, multiplying the remain-

ing number by 4, we obtain 305500, which is the quotient sought.

In the same manner, to divide 678375000 by 125, it is only necessary to cut off the 3 ciphers, and to multiply the remaining number by 8; the result is 5427000, which is the quotient sought.

But if the dividend do not terminate in ciphers, the process is scarcely less simple. Let it be proposed to divide 634782 by 25: we shall first subtract from it the last 2 digits, resolving it into the following parts: —

634700

82

Each of these must be divided by 25. The first may be so divided by cutting off the ciphers and multiplying the remaining number by 4: the product, 25388, is a part of the quotient sought. The second number above must now be divided by 25; but that number being less than 100, its quotient by 25 is always readily perceived. In the present case 25 is contained 3 times in 82, with a remainder 7. This quotient 3 must be added to the former, 25388, and the total quotient is 25391, the remainder being 7.

In like manner, if it were required to divide 634782 by 125, the process would be as follows: —

Multiply	634000	•	125) 782 (6
By	<u>4</u>		<u>750</u>
	2536		Remainder <u>32</u>
Add	<u>6</u>		
Quotient	2542		

It is evident that this process consists in resolving the dividend into two parts, and dividing these parts separately by 125, by different methods, the partial quotients being added together to obtain the total quotient.

A like method may be extended to any divisor which is a sub-multiple of 100, 1000, 10000, &c.

(189.) We have already explained several methods of proving Division by Multiplication (167. et seq.); but

when the divisor and dividend consist of several places, these methods are often tedious, and it is desirable that we should possess some short means of checking arithmetical computations of this kind. The method of casting out *the nines*, already explained (161.), may be used for this purpose. It has been shown, that if the remainder, after the process of division has been performed, be subtracted from the dividend, we shall obtain a number which, if the work be correct, should be the product of the divisor and quotient. Whether it is so or not may be determined by applying to it the method of casting out the nines, considering it as the result of the multiplication of the divisor and quotient. This is, perhaps, the shortest method of verification which can be used for questions in Division. It is, however, as in Multiplication, liable to failure, though in rare cases.

(190.) The process of division may be used as a means of verifying that of multiplication, if the product obtained in multiplication be taken as the dividend, and either the multiplier or multiplicand as divisor. The quotient obtained should be the multiplicand when the multiplier is taken as divisor, and the multiplier when the multiplicand is taken as divisor. The teacher may check the work of his pupils by proposing to one pupil, as a question in Division, the result obtained by another in Multiplication.

(191.) After what has been explained in the present chapter, the following general directions for the solution of questions in Division will be easily understood.

GENERAL RULES.

1. *Write the divisor on the left of the dividend, separated from it by a line : place another line on the right of the dividend after the units' place, to separate the quotient from the dividend; the quotient being afterwards written on the right of that line.*

2. *Count off from the left of the dividend, or from its highest place, as many digits as there are places in the divisor : if the number formed by these be less than the divisor*

then count off one more ; consider these digits as forming one number, and find how often the divisor is contained in that number : it will be always contained in it less than 10 times, as we shall presently perceive, and therefore the quotient of the division will always be a single digit : place this single digit as the first figure of the quotient.

3. *Multiply the divisor by the same digit, and place the product under those figures of the dividend which were taken off on the left, and then subtract such product from the number above it, by which you will obtain the first remainder : this remainder must be less than the divisor, for the digit, placed as the first figure of the quotient, expressed the greatest number of times which the divisor was contained in the number cut off from the dividend. If the remainder were greater than the divisor, this would not be the case.*

4. *On the right of the first remainder place that figure of the dividend which next succeeds those which were cut off to the left : find how often the divisor is contained in the number thus formed. It will be contained in it less than 10 times, and therefore the quotient will be a single digit : place this digit as the next figure of the quotient.*

5. *Multiply the divisor by the same digit, and place the product under the first remainder, with the digit of the dividend annexed : subtract it from that number, and you will obtain the second remainder.*

6. *On the right of the second remainder place the succeeding figure of the dividend, and proceed with the number thus formed as in the former case, and the third digit of the quotient will be obtained.*

7. *Continue to annex the succeeding figures of the dividend to the succeeding remainders until every figure of the dividend has been thus brought down : the division will then be complete ; and the last remainder being subtracted from the dividend would leave a remainder, which should be equal to the product of the quotient and divisor.*

It has been stated above, that if a digit be annexed to the right of a number less than the divisor, the divi-

nor will be contained in that number less than 10 times. To perceive the truth of this, we have only to consider that when the digit is so annexed to the preceding figures of the number, forming a number less than the divisor, we should obtain a greater number by annexing a 0 to the divisor than we could by annexing any digit to the number in question. This will be easily understood when applied to an example. Let the divisor be 1534, and let the number which is less than the divisor, and to which a digit is to be annexed, be 1527 : the greatest digit which can be annexed to this is 9, by which it will become 15279 : this will, evidently, be less than 10 times the divisor, because, if we annex a 0 to the latter, we shall obtain a number consisting of the same number of places, but of which the first 4 places form a greater number. Since, then, 10 times the divisor is greater than the number found by annexing the digit, the divisor must be contained in that number less than 10 times.

If the first or any succeeding remainder, with the digit brought down from the dividend annexed, form a number less than the divisor, then a cipher must be written in the quotient, and another figure brought down. If the number thus obtained be still less than the divisor, another 0 must be written in the quotient, and the next figure of the dividend brought down ; and this must be continued until a number is obtained greater than the divisor. If, after all the figures of the dividend are brought down, the number should still be less than the divisor, then that number must be taken as the remainder in the division, and the quotient will be a number terminating in ciphers.

(192.) When there is no remainder in the process of division, if the divisor and dividend be both multiplied by the same number, the quotient will remain unaltered. This follows immediately from what has been already proved in Multiplication, that when the multiplicand is increased any number of times, the multiplier remaining the same, the product will be increased the same

number of times. Since the dividend must be the product of the divisor and quotient, it follows, that if we increase the divisor any number of times, the quotient remaining the same, we must increase the dividend the same number of times. It is evident, that if the divisor be contained in the dividend 6 times, 10 times the divisor will be contained in 10 times the dividend also 6 times.

(193.) We may hence infer also, that if the divisor and dividend be both divided by the same number, the quotient will remain the same; since this may be considered as merely doing away with a previous multiplication of both by the same number. If the divisor be contained 6 times in the dividend, the tenth part of the divisor will be contained 6 times in the tenth part of the dividend.

(194.) From this principle we may derive a method of abridging the process of division in the case where the divisor and dividend both terminate in ciphers. We may in such case cut off the same numbers of ciphers from both, because by so doing we divide them both by the same number, such as 10, 100, 1000, &c. according to the number of ciphers cut off.

(195.) In general, if it be apparent on inspection that both divisor and dividend are divisible by any number, the process of division may be abridged by previously dividing both of them by that number. Thus, if we would divide 72 by 24, we may previously divide each of them by 8, and the question will be reduced to the division of 9 by 3, the quotient of which is 3.

(196.) When the dividend is not an exact multiple of the divisor, and there is therefore a remainder, we may still multiply the divisor and dividend by the same number without altering the quotient. But it will be necessary in this case to multiply the remainder by that number. This will be apparent from considering that the remainder constitutes part of the dividend; and therefore the multiplication of the dividend by the proposed

number necessarily infers the multiplication of the remainder by that number.

For example: if we divide 79 by 8, we get the quotient 9, with a remainder 7. Let us now multiply the divisor, dividend, and remainder by 10, and we shall find, that by dividing 790 by 80, we shall still get the quotient 9, but will have the remainder 70; and a similar result would be obtained by whatever number we might multiply the divisor and dividend.

(197.) By reversing this process, it follows, that we may divide the divisor and dividend by the same number without altering the quotient; but that having done so, the remainder which will be obtained in the division will be as many times less than the remainder which would be obtained had the divisor and dividend remained unaltered, as the original divisor and dividend are greater than those obtained by division.

In general, then, the process of division may be abridged, when the divisor and dividend are both obviously divisible by the same number without a remainder. We may in that case divide them by the same number, and operate on the quotients thus obtained by the usual process of division. It will only be necessary to multiply the remainder by that number by which the original divisor and dividend were divided.

BOOK II.

FRACTIONS.

CHAPTER I.

ON THE LANGUAGE AND NOTATION OF FRACTIONS. — VARIOUS
WAYS OF EXPRESSING THEM. — THEIR RELATIVE VALUES. —
THEIR ADDITION AND SUBTRACTION.

(198.) **W**E have seen that, except in the particular case in which the dividend happens to be an exact multiple of the divisor, it is impracticable, by means of the language and notation of number hitherto explained, to complete the process of division. When the dividend, as generally happens, is not an exact multiple of the divisor, the quotient which we obtain is not the actual quotient of the dividend divided by the divisor; because the remainder, being a part of that dividend, has not been so divided. The quotient found in this case is, in fact, the true quotient which would be obtained by using as a dividend not the actual dividend, but the number which would be obtained by subtracting the remainder from the proposed dividend. To complete the process, and obtain the true quotient, it would be necessary to divide the remainder by the divisor: but the remainder being less than the divisor, the ordinary process of division, so far as we have yet explained that operation, becomes inapplicable.

It will be recollected that the process of division was presented under two distinct points of view: first, as that operation by which it is discovered how many times the divisor is contained in the dividend; and secondly, as the process by which the dividend is resolved into as many equal parts as there are units in the divisor

Now, according to the first view of this operation, it is clearly impracticable when the divisor is greater than the dividend; for in that case the divisor is not contained in the dividend at all. Under the second aspect, however, we arrive at a somewhat different conclusion. When the dividend is less than the divisor, if we were to attempt to divide the former into as many equal parts as there are units in the divisor, it is evident that the number of those parts would be greater than the number of units which the dividend itself contains, and consequently each of the parts must be less than a unit. So far as the language and notation of number hitherto explained go, they are incapable of expressing any thing less than the unit; but there is evidently no absurdity or difficulty in conceiving quantities less in any degree than the unit, and therefore the absence of proper means of expressing such quantities is a defect in the nomenclature of number which must be removed.

If, instead of using them in the abstract, we apply numbers to express any particular species of quantity, we shall have no difficulty in perceiving the necessity of providing means for expressing numbers less than the unit. Let us suppose that the dividend in any question of Division expresses a certain number of inches, which are to be divided into as many equal parts as there are units in the divisor. Now, suppose it so to happen that, the divisor being 17, there is a remainder 10 inches: to complete the division it would be necessary that these 10 inches should be divided by 17; that is, that they should be divided into 17 equal parts. Now, whatever difficulty we may have in conceiving the division of 10 abstract units into 17 equal parts, we certainly can have none whatever in conceiving the division of a line 10 inches in length into 17 equal parts. It will be evident that each of these parts is less than an inch, the whole 17 of them making up 10 inches. We shall now proceed to explain the system of language and notation by which quantities which are either less than the unit,

or which are not an exact multiple of unity, may be expressed by numbers.

(199.) Such numbers are called **FRACTIONS**; while the name **INTEGERS**, or **WHOLE NUMBERS**, is applied to unity and its multiples.

(200.) Recollecting that the unit of number in its practical application always expresses some quantity which admits of subdivision, let us conceive the unit divided into any proposed number of equal parts, as 10 : each of these parts is called a *tenth*, meaning a tenth part of the unit. In like manner, if the unit be supposed to be divided into 8 equal parts, each part is called an *eighth* ; if into 9 equal parts, a *ninth* ; and so on.

The unit being always a multiple of such parts, these parts are called *sub-multiples* of the unit.

(201.) We have frequently occasion to express two or more of those parts into which the unit is supposed to be divided. In such cases we first name the number of parts to be expressed, and next the number of parts into which the unit is divided. Thus, if we suppose the unit divided into 10 equal parts, and that we wish to express 7 of these parts, we call them *seven tenths* : if the unit be supposed to be divided into 9 equal parts and that we wish to express 5 of them, we call them *five ninths* ; and so on. •

(202.) In the case of the remainders in incomplete divisions, a number greater than unity is generally required to be divided into a greater number of parts than the units it contains. Thus, if 5 be the remainder and 9 the divisor, to complete the division it would be necessary to divide 5 into 9 equal parts. To effect this, let us imagine each of the units which compose the number 5 to be divided into 9 equal parts. It is evident that by taking the ninth part of each unit, and adding such parts together, we shall obtain the ninth part of the whole : we shall thus have five ninth parts of a single unit ; from whence it appears that five ninths of a single unit is the same as the ninth part of five units. In the

same manner, if it be required to divide 10 into 13 equal parts, we should imagine each of the units which compose 10 to be divided into 13 equal parts, and one of such parts taken from each unit to make up the thirteenth part of the whole: the thirteenth part, therefore, of 10 is the same as ten thirteenth parts of the unit.

(203.) From what has been here explained, it will be perceived that in order to express a fraction two numbers are necessary,—one, which expresses the number of equal parts into which the unit is supposed to be divided; and the other, the number of those parts intended to be taken in the fraction. That which expresses the number of parts into which the unit is supposed to be divided is called the **DENOMINATOR** of the fraction; and that which expresses the number of those parts which compose the fraction is called its **NUMERATOR**. Thus, if the fraction be five ninths, the unit is supposed to be divided into nine equal parts, five of which compose the fraction: therefore the denominator is in this case nine, and the numerator five. In the same manner if the fraction be ten thirteenths, the unit is supposed to be divided into thirteen equal parts, and ten of these compose the fraction: the denominator is, therefore, thirteen, and the numerator ten.

Such is the nomenclature by which numbers, which are either less than the unit, or which are not an exact multiple of that unit, are expressed. We shall now explain the notation by which the same are expressed by the aid of figures.

(204.) The figures expressing the numerator are usually placed above a line, and those expressing the denominator below it. Thus the fractions five ninths and ten thirteenths are expressed in figures in the following manner,—

$$\frac{5}{9} \qquad \frac{10}{13}$$

(205.) It is not necessary that the language and notation of fractions, here explained, should be confined to numbers less than the unit. On the contrary, in practical calculations it is often convenient, if not ne-

cessary, to express fractions not only greater than the unit, but greater than numbers of more considerable magnitude. There is no difficulty in conceiving the application of the nomenclature just explained to such numbers. Having supposed the unit divided into any number of parts, we can easily conceive those parts repeated much oftener than the number of times that they are contained in the unit. Thus, if an inch be divided into 10 equal parts, it is as easy for us to conceive 11 of these parts as 9 of them ; and we can go on increasing their number to any extent whatever : thus 35 or 99 tenths of an inch is just as intelligible as 35 or 99 inches. It is true that such numbers may be expressed otherwise and more briefly ; for since every 10 tenths make an inch, 35 tenths will be 3 inches and 5 tenths, and 99 tenths will be 9 inches and 9 tenths. Nevertheless it is often more convenient to express them in the purely fractional form.

(206.) Fractions greater than the unit are frequently distinguished from those less than the unit by the name *improper* fractions, — a proper fraction being one which is less than the unit, and all others being called improper.

(207.) From what has been explained above, it will appear that a fraction may be considered under two points of view ; first, either as a certain number of sub-multiples of the unit ; or, secondly, as a quotient whose dividend is the numerator, and divisor the denominator.

(208.) It follows also that, by the aid of the notation of fractions, we can at once render complete all those processes of division explained in the last chapter in which there is a remainder. It is only necessary to add to the integral part of the quotient a fraction whose numerator is the remainder, and whose denominator is the divisor, such a fraction expressing the quotient arising from the division of the remainder by the divisor. Thus, if we would divide 11 by 3, we shall find the integral part of the quotient to be 3, with a remainder 2 ; this remainder being divided by 3, gives the quotient

, and therefore the total quotient is $3\frac{2}{3}$, which means 3 and 2 thirds.

(209.) One integer, or whole number, may be considered as a fraction of another, which, in that case, relatively to it is taken as the unit. Thus we say, that 3 is the fourth part of 12, that 9 is 3 fourths of 12, that 4 is the fifth of 20, that 3 is 3 fifths of 5, and so on.

(210.) The numerator of a fraction is always the same fraction of its denominator as the fraction itself is of the unit. This is evident from the explanations already given. In the fraction $\frac{3}{4}$ the numerator is 3 fourths of the denominator; and the fraction itself is 3 fourths of the unit. In the same manner in the fraction $\frac{5}{6}$, the numerator is 5 sixths of the denominator, and the fraction itself 5 sixths of the unit. Since there is an unlimited variety of whole numbers, which are the same fractions of other whole numbers, it follows that the same fractional numbers may be expressed in an unlimited variety of ways. Thus, for example, since 3, 6, and 9 are respectively 3 fourths of 4, 8, and 12, the fractions $\frac{3}{4}$, $\frac{6}{8}$, $\frac{9}{12}$ are equal; because the numerator of these being 3 fourths of its denominator, the fractions must each be 3 fourths of the unit. Let us consider how this may be reconciled with the nature of fractions above explained. In the fraction $\frac{6}{8}$ the numerator and denominator are respectively double the numerator and denominator of $\frac{3}{4}$: when we double the denominator of $\frac{3}{4}$, we double the number of parts into which the unit is divided, and each part will therefore have only half its former magnitude. To make up the same amount it would be necessary, therefore, to take twice the former number of parts; but that is done by doubling the numerator. If we increased the denominator alone in a twofold proportion, we should diminish the magnitude of the parts composing the fraction in that proportion. If we increased the numerator alone in a twofold proportion, we should double the number of parts composing the fraction without changing the magnitude of those parts. By the one process we should

reduce it to half its former magnitude; and by the other, we should increase it to double its former magnitude. When both these operations, however, are performed at once, the two effects neutralise each other, and the magnitude of the fraction remains unaltered.

Let us apply this reasoning to an example. — Take a line a foot long: if we divide it into 4 equal parts, and take 3 of these, we shall have 3 fourths of a foot; but if we divide the same line into 8 equal parts, each of these parts will be only half the magnitude of the former ones: 3 of them, therefore, would be half the length of 3 fourths of a foot; and 6 of them would, consequently, be equal to 3 fourths of a foot. Thus, $\frac{3}{4}$ of a foot and $\frac{6}{8}$ of a foot are the same length. In the same manner it may be shown, that $\frac{9}{12}$ of a foot is also the same length.

(211.) Since the numerator and denominator of a fraction may be multiplied by the same number without changing the value of the fraction, they may also be divided by the same number without changing its value; for this is only undoing the previous multiplication. Thus, $\frac{6}{9}$ is reduced to $\frac{2}{3}$ by dividing its numerator and denominator by 3, and these fractions have evidently the same value.

From all that has been just stated, it appears that the value of a fraction does not depend on the absolute magnitudes of the numbers forming its numerator and denominator, since these magnitudes may be subject to unlimited variation; the value of the fraction remaining unchanged. The value of the fraction, however, depends on the *relative or proportional* magnitudes of its numerator and denominator. So long as the numerator and denominator retain the same relative or proportional values, so long will the fraction remain unaltered in its magnitude. The following fractions, differing very much in the magnitude of their numerators and denominators, are, nevertheless, themselves of the same magnitude: —

$$\frac{2}{3} \quad \frac{19}{27} \quad \frac{15}{18} \quad \frac{20}{25} \quad \frac{30}{36} \quad \frac{40}{48}$$

In the same manner, the following are of equal value:—

$$\frac{5}{7} \quad \frac{10}{14} \quad \frac{15}{21} \quad \frac{20}{28} \quad \frac{25}{35} \quad \frac{30}{42} \quad \frac{35}{49}$$

In this series each succeeding form of the fraction is found by multiplying the numerator and denominator of the first form by the same number; and, on the other hand, the first form in each series may be found from any of the others, by dividing both numerator and denominator by the same number.

(212.) The numerator and denominator of a fraction are called its *terms*.

(213.) From what has been stated, it appears that the *terms* of a fraction may be increased without limit; because there is no limit to the magnitude of the number by which we may multiply them. But since they may not be both capable of being divided by the same number, or, if so, that number may be limited in magnitude, there is a limit to the extent to which its terms may be diminished.—Take the example $\frac{1}{2} \frac{5}{10}$. We may multiply both terms of this fraction by any number, however great, and we shall get an equivalent fraction whose terms are proportionally great; but the same terms cannot be divided by any number greater than 5, and, consequently, the terms of the fraction cannot be reduced lower than $\frac{1}{5}$. When the terms of a fraction are thus divided by the greatest number which exactly divides them both, the fraction is therefore reduced to its *least* or *lowest terms*.

Owing to the convenience of using small numbers, fractions are generally expressed in their lowest terms, unless in particular cases, which will be explained, hereafter. Thus, we do not commonly use the fraction $\frac{300}{400}$, but the equivalent fraction $\frac{3}{4}$, which is in its lowest terms.

(214.) From what has been already explained, it appears that, provided the denominator of a fraction remain unaltered, the fraction will be increased or diminished in whatever proportion its numerator is increased or diminished (210). Hence a fraction may always be mul-

multiplied or divided by a whole number, by multiplying or dividing its numerator by that number: thus, if we would divide $\frac{4}{9}$ by 2, we must divide its numerator by 2, preserving the same denominator, and the quotient is $\frac{2}{9}$. If we would, on the other hand, multiply the fraction by 2, we must multiply its numerator by 2, and the product is $\frac{8}{9}$. It is evident that half of 4 ninths is 2 ninths, and twice 4 ninths is 8 ninths.

(215.) Since it has been proved that in whatever proportion the denominator is increased the fraction is diminished, and in whatever proportion the denominator is diminished the fraction is increased, it follows that by multiplying the denominator by any number we divide the fraction by the same number, and by dividing the denominator by any number we multiply the fraction by that number. — Let the fraction, for example, be $\frac{3}{16}$. If we multiply the denominator by 2, it becomes $\frac{3}{32}$. Now, since a sixteenth part of the unit is half of an eighth part, 3 eighths must be double 3 sixteenths; that is, $\frac{3}{16}$ is half of $\frac{3}{8}$. Thus, by doubling the denominator of $\frac{3}{16}$ we actually divide the fraction by 2. Let us next suppose that we divide the denominator of $\frac{3}{8}$ by 2, and obtain $\frac{3}{4}$. We have here really multiplied the fraction by 2; for since a fourth of the unit is twice the magnitude of an eighth, 3 fourths will be twice the magnitude of 3 eighths; and, therefore, $\frac{3}{4}$ is double $\frac{3}{8}$.

When fractions occur in arithmetical calculations in terms which are not their lowest, it is often required to reduce them to their lowest terms. To accomplish this, it is necessary to find the greatest number which divides exactly both numerator and denominator.

(216.) A number which divides exactly 2 or more other numbers without a remainder is called a *common measure* of these numbers; and the greatest number which divides 2 or more others without a remainder is called the *greatest common measure* of these others. Thus, 2 is a common measure of 8 and 12, because it divides both without any remainder; but it is not the

greatest common measure of 8 and 12, because 4 also divides 8 and 12 without any remainders. No number greater than 4 divides 8 and 12; and, therefore, 4 is their greatest common measure. If it be required to reduce the fraction $\frac{4}{12}$ to its lowest terms, we must divide its numerator and denominator by their greatest common measure 4; after which the fraction will become $\frac{1}{3}$, which is evidently in its lowest terms, since no other whole number divides 2 and 3.

(217.) It has been already explained, that a prime number is one which has no divisor save the unit. In like manner, 2 numbers are said to be *relatively prime*, or *prime to each other*, when they have no common measure greater than the unit. Thus, 7 and 9 are prime to each other, although the latter is not a prime number, being measured by 3.

Two numbers may be prime to each other although neither by itself is prime. Thus, 8 and 9 are relatively prime, although neither of them is a prime number.

If either of 2 numbers be absolutely a prime number, they must be prime to each other; for since that which is absolutely prime is not measured by any other number greater than the unit, the 2 numbers can have no common measure.

(218.) From these circumstances we may infer, that whenever either of the terms of the fraction is a prime number, the fraction is in its least terms; and a fraction is always in its least terms when its terms are relatively prime.

(219.) Two even numbers cannot be relatively prime, since 2 is always a common measure of them: hence a fraction, having both its terms even, can be reduced to lower terms by dividing the numerator and denominator by 2.

(220.) If a number ends in either 5 or 0, it is always divisible by 5 without a remainder: hence, if both terms of a fraction end in 5 or 0, or one in 5 and the other in 0, the fraction may be reduced to lower terms by dividing both numerator and denominator by 5.

(221.) If both terms of a fraction terminate in ciphers, the same number of those ciphers may be struck off from both ; for, by so doing, both terms of the fraction are divided by 10, 100, 1000, &c. according to the number of ciphers struck off.

(222.) If the digits of a number added together give a sum which is an exact multiple of 9, the number itself will then be an exact multiple of 9 (161 note). If this happen with both terms of a fraction, we may infer that they are both divisible by 9, and in this way the fraction may be reduced to lower terms. Thus, in the fraction $\frac{5788}{6837}$, we find that the digits of the numerator added together make 27, which is a multiple of 9 ; and the digits of the denominator added together make 18, which is also a multiple of 9. We therefore infer that the numerator and denominator may both be divided by 9 without a remainder : performing this division, we find the quotient to be 532 and 759. The proposed fraction is, therefore, equivalent to $\frac{532}{759}$.

(223.) This, and such methods of reduction, will serve, however, only in particular cases. To obtain a general method of reducing a fraction to its lowest terms, it is necessary that we should be able to find the greatest common measure of its numerator and denominator.

Let the numerator and denominator be 376 and 788. To find the greatest common measure of these, we shall proceed in the following manner : —

$$\begin{array}{rcl}
 & \text{B} & \text{A} \\
 376 &) & 788 \quad (2 \\
 \text{C} & - & 752 \\
 \hline
 & \text{D} & - 36 \quad) 376 \quad (10 \\
 & \text{E} & - 360 \\
 \hline
 & \text{F} & - 16 \quad) 36 \quad (2 \\
 & \text{G} & - 32 \\
 \hline
 & \text{H} & - 4 \quad) 16 \quad (4 \\
 & & 16
 \end{array}$$

In explaining the process here written, we shall, for brevity, express the different numbers by the letters pre-

fixed to them. Of the 2 numbers A and B, whose common measure is to be found, we divide the greater A by the lesser B; and we find the quotient 2, with the remainder D. We then take B for the dividend, and D for divisor, and make another division, where we find the quotient 10, and the remainder F. Again, taking D for the dividend, and F for the divisor, we make another division, in which we get the quotient 2, and the remainder H. Taking H in like manner as divisor, and F as dividend, we get the quotient 4, without a remainder: *the last remainder, H, is the greatest common measure of the proposed numbers A and B.*

We may prove that the number H is a common measure of A and B in the following manner:—By the last division, it appears that H measures F four times, and by the preceding division it appears that G is twice F. Since H measures F four times, it will therefore measure twice F (that is G) eight times. Now, D is obtained by adding G and H together; and since H measures G eight times, it measures $G + H$ nine times; that is, it measures D nine times. By the second division it appears that E is ten times D; and since H measures D nine times, it measures E ninety times; but it measures F four times, and, consequently, it measures $E + F$ ninety-four times. Now, B is obtained by adding E to F; and, consequently, H measures B ninety-four times. Now, since H measures B ninety-four times, it must measure twice B (that is C) twice ninety-four times, that is, 188 times; but it measures D nine times: therefore it measures $C + D$, that is, A, 197 times. It follows, then, that H measures B 94 times, and A 197 times; and, consequently, is a common measure of A and B.

We shall now show that H is the greatest common measure of A and B, by proving that if A and B have any other common measure, that common measure must measure H, and must be therefore less than it.

If such a common measure exist, since it measures B it measures C, which is twice B; and since it mea-

asures A and C, it must measure their difference D. This will be easily understood by an example.—Thus, if it measured A ten times and C eight times, it would measure D twice. Since, then, it measures D it must measure E, which is ten times D; and since it measures B and E, it must, as before, measure their difference F. In like manner, since it measures F it must measure twice F, which is G; and since it measures D and G, it measures their difference H, and is therefore less than H. Hence, if A and B have any other common measure besides H, that common measure must measure H, and therefore be less than it. By generalising the above process, we shall find a rule for obtaining the greatest common measure of two numbers.

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Divide the greater by the less, and divide the less by the remainder. If there be any remainder, divide this first remainder by the next remainder; and so continue dividing each remainder by the following remainder, until a remainder is found which exactly divides the preceding remainder. Such a remainder is the greatest common measure of the two numbers.

If the last remainder obtained by this process happen to be 1, then 1 is the greatest common measure of the two proposed numbers, and they are therefore relatively prime, and the fraction whose numerator and denominator they are is in its least terms.

(224.) Since a fraction may always be regarded as a mode of expressing the quotient of the numerator divided by the denominator, it follows that an improper fraction may always be reduced to a number partly integral and partly fractional, called a *mixed number*, by performing the process of division by the rule given in the last chapter. In that case the integral part of the number will be the quotient of the incomplete division, and the fractional part will be a proper fraction, whose

numerator is the remainder, and whose denominator is the divisor. This will be easily understood from an example.—Let the improper fraction be $\frac{97}{31}$: the quotient in the process of division would be 3, and the remainder 4; but, to make the division complete, it would be necessary to divide this remainder 4 by 31, and add the quotient to the 3 already found; but the quotient of 4 by 31 is expressed $\frac{4}{31}$, and therefore the complete quotient of the division is $3\frac{4}{31}$.

(225.) An improper fraction may happen to be equivalent to an integer. This will occur whenever the numerator is a multiple of the denominator. Thus, the fraction $\frac{36}{4}$ is equivalent to the integer 9.

(226.) Hence it is apparent, that an integer may always be converted into an improper fraction having any proposed denominator, by multiplying the integer by the proposed denominator, and subscribing the same denominator below it. Thus, 9 may be converted into an improper fraction whose denominator is 4, by multiplying it by 4, and writing 4 under it, $\frac{36}{4}$.

(227.) The number which stands between proper and improper fractions is a fraction whose numerator and denominator are the same number, and such a fraction is obviously equal to the unit. If we regard it as a fraction, we consider the unit to be divided into a certain number of parts, and the fraction to consist of the same number of those parts. If we regard it as a quotient, the divisor is contained in the dividend once without a remainder.

It is sometimes necessary to consider the unit under such a point of view, as we shall see hereafter.

(228.) Since the relative magnitudes of fractions depend upon two numbers, while those of integers depend on only one, the latter is much more evident on inspection than the former. It is not always easy at first view to pronounce which of two fractions is the greater, especially if their numerators and denominators are high numbers. It is, therefore, of considerable importance to possess the means of at once determining

the relative amount or value of any two fractions which can be proposed.

The difficulty which attends the perception of the comparative value of fractions is one which attends all quantities which are expressed by numbers having different units. Let us suppose that two sums of money are expressed, one in pounds and the other in shillings; and that we are either required to declare which is the greater, or to express their proportional value, or to add or subtract them, so as to express their sum or difference by a single number. We should find it impossible to do so, unless the numbers in question were first submitted to some change, so as to render them directly comparable, or capable of addition or subtraction. Let the two numbers proposed be 479 shillings and 23 pounds. We are required to declare the proportional value of these sums, or to add them together so as to express their amount by a single number. It is evident that their addition would not be effected by adding 23 to 479; for in that case the sum, which would be 502, would neither express shillings nor pounds, some of its units being of one kind and some of another. In order to enable us to compare the values of the two sums, or to add or subtract them so as to express their sum or difference by a single number, we must, in the first instance, effect such a change in one or both of them *that their component units shall be the same*. This may be done by discovering the number of shillings which are contained in 23 pounds. Since there are 20 shillings in a pound, there are 23 times that number in 23 pounds: hence 23 pounds are equivalent to 460 shillings. We can now express both sums in shillings, the one being 460 and the other 479. It is manifest that their comparative values are expressed by these numbers; also, that by adding these numbers we may get a single number, 939, which shall express the number of shillings in their sum, and by subtracting the lesser from the greater we shall get a number, 19,

which will express the number of shillings in their difference.

From this example it will be apparent that, in cases where two numbers composed of different units are required to be compared, added or subtracted, they must be previously, by some means, converted into other equivalent numbers which are composed of the *same units*.

(229.) The operation by which a number composed of units of one kind is converted into another equivalent number composed of units of another kind is called, in arithmetic, *Reduction*. Such, for example, is the operation by which a number whose units are pounds is converted into an equivalent number whose units are shillings.

We have already stated, that a fraction may be regarded as a whole number if the parts of which it consists be considered as units. Thus, $\frac{3}{4}$ may be expressed by the whole number 3, if it be at the same time declared that its component units are *fourths*. Considered in this point of view, fractions having the same denominator are numbers composed of the same units, and may be expressed by their numerators taken as whole numbers, if the value of their component units be at the same time declared. Thus, for example, the fractions $\frac{9}{12}$, $\frac{7}{12}$, $\frac{5}{12}$, $\frac{3}{12}$ may be expressed as follows:—

twelfths	twelfths	twelfths	twelfths
9	7	5	3

The relative magnitudes of these fractions will obviously be expressed by their numerators: their total amount would be obtained by adding those numerators, still bearing in mind that the units they express are twelfths; and any one of them may be subtracted from another in the same manner.

(230.) But if, instead of writing over the numerator the nature of the unit of which the number is composed, we subscribe in the usual way the denominator, then the same operations may be still performed in the same

manner, the denominator being written under the result, to signify the magnitude of the units of which the number is composed. From these considerations we shall deduce the following general conclusions : —

1. Fractions having the same denominator have the same relative magnitudes as their numerators.

2. Fractions having the same denominator may be added by adding their numerators, still preserving the same denominator ; thus, $\frac{0}{2} + \frac{7}{2} = \frac{7}{2}$.

3. Fractions having the same denominator may be subtracted by subtracting their numerators, still preserving the same denominators ; thus, $\frac{0}{2} - \frac{7}{2} = \frac{-7}{2}$.

(231.) Since, then, the addition and subtraction of fractions can only be performed, and their relative values expressed, when they have the same denominator, it is necessary, in order to be enabled to perform these operations, and to express the relative magnitude of fractions, that we should possess some means of *converting any proposed fractions having different denominators into other equivalent fractions having the same denominator*.

We have already shown that the same fraction may be expressed in a great variety of forms, either by multiplying or dividing its numerator and denominator by the same number. Now, if two fractions have different denominators, let us suppose all the possible fractions equivalent to each of them with other numerators and denominators found : among these may be discovered two that have the same denominator ; the numerators of these two will then express the relative values of the two fractions, and either addition or subtraction may be performed by the rule just given. Let the two fractions $\frac{2}{3}$ and $\frac{4}{6}$ be proposed, and let it be required to discover 2 fractions equivalent to these having the same denominator. Since the terms of neither of these fractions admit of being both divided by any number, we shall find all the equivalent fractions by multiplying their numerators and denominators successively by 2, 3, 4, &c. They will be as follows : —

$$\frac{4}{5} = \begin{cases} \frac{4 \times 2}{5 \times 2} = \frac{8}{10} & \frac{4 \times 3}{5 \times 3} = \frac{12}{15} & \frac{4 \times 4}{5 \times 4} = \frac{16}{20} \\ \frac{4 \times 5}{5 \times 5} = \frac{20}{25} & \frac{4 \times 6}{5 \times 6} = \frac{24}{30} & \frac{4 \times 7}{5 \times 7} = \frac{28}{35}, \text{ \&c.} \end{cases}$$

$$\frac{5}{6} = \begin{cases} \frac{5 \times 2}{6 \times 2} = \frac{10}{12} & \frac{5 \times 3}{6 \times 3} = \frac{15}{18} & \frac{5 \times 4}{6 \times 4} = \frac{20}{24} \\ \frac{5 \times 5}{6 \times 5} = \frac{25}{30} & \frac{5 \times 6}{6 \times 6} = \frac{30}{36}, \text{ \&c. \&c.} \end{cases}$$

On examining this series of equivalent fractions, we find that $\frac{8}{10}$ and $\frac{20}{25}$ have the same denominator. These equivalent fractions have been obtained by multiplying the numerator and denominator of the first of the proposed fractions by 6, and those of the second by 5. Now, it will be observed, that 6 is the denominator of the second, and 5 of the first; and we therefore infer that this reduction has been made by multiplying both terms of the first fraction by the denominator of the second, and both terms of the second by the denominator of the first.

Let us now enquire whether this method is general in its nature, or accidental on the particular fractions taken in the above example. Let the proposed fractions be $\frac{8}{9}$ and $\frac{6}{7}$. If we multiply both terms of the first by 7, the denominator of the new fraction will be the product of 9 and 7; and if we multiply both terms of the second by 9, the denominator of the new fraction will be, in like manner, the product of 9 and 7. In this case, therefore, also the equivalent fraction obtained in this way will have the same denominator. But, in general, if we multiply both terms of the first fraction by the denominator of the second, the new denominator will evidently be the product of the denominators of the proposed fractions; and, in like manner, if we multiply both terms of the second by the denominator of the first, the denominator of the fraction so obtained will be likewise the product of the denominators of the proposed fractions. Thus we perceive that the application of this principle will, in general, enable us, whenever two fractions are proposed with different denominators, to find

two other fractions equivalent to those with the same denominator.

(232.) When it is required to convert three or more fractions having different denominations into equivalent fractions having the same denominator, it is only necessary first to apply the principle just explained to any two of them, and then to repeat its application to these two and a third, and so on. The practical process may, however, be abridged by observing the following

RULE.

Multiply both terms of each fraction by the continued product of the denominators of the others, and equivalent fractions will be obtained having the same denominator, which denominator will be the continued product of the denominators of the proposed fractions.

For example, let the proposed fractions be $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$; multiply both terms of the first by $4 \times 5 \times 6$; multiply both terms of the second by $3 \times 5 \times 6$; both terms of the third by $3 \times 4 \times 6$; and both terms of the fourth by $3 \times 4 \times 5$. We shall obtain the following equivalent fractions:—

$$\begin{array}{lcl} \frac{2 \times 4 \times 5 \times 6}{3 \times 4 \times 5 \times 6} = \frac{240}{360} & \frac{3 \times 3 \times 5 \times 6}{3 \times 4 \times 5 \times 6} = \frac{270}{360} \\ \frac{4 \times 3 \times 1 \times 6}{3 \times 4 \times 5 \times 6} = \frac{360}{360} & \frac{5 \times 3 \times 1 \times 5}{3 \times 4 \times 5 \times 6} = \frac{300}{360}. \end{array}$$

It will be perceived that, in these equivalent fractions, the common denominator is the continued product of the denominators of the 4 fractions proposed; and also, that the numerators of each are found by multiplying the numerator of each fraction by the continued product of the denominators of the other fractions.

Though not indispensably necessary, yet it will be found expedient, before we proceed to reduce fractions to a common denominator, to reduce them to their lowest terms, since otherwise we shall have to deal with higher numbers in the calculation than would be necessary.

(233.) To find numbers which would express the

relative values of any proposed fractions, it is only necessary to find the numerators of equivalent fractions having the same denominator. These numerators are, as we have seen above, the products of the numerator of each fraction, and the denominator of the other. Thus, if the fractions proposed be $\frac{2}{3}$ and $\frac{3}{4}$, the numerators of the equivalent fractions would be 2×4 and 3×3 ; and, consequently, the two fractions will be in the proportion of 8 to 9. In the same manner, if the fractions proposed were $\frac{5}{6}$ and $\frac{7}{8}$, the numerators of the equivalents would be 5×8 and 7×6 ; and the proportion of the proposed fractions would be that of 40 and 42.

(234.) From all that has been explained, the truth of the following rule for the addition and subtraction of fractions will be apparent.

RULE.

To add or subtract fractions.

1. *If they have the same denominator, add or subtract their numerators, and subscribe their denominator.*
2. *If they have not the same denominator, find equivalent fractions which have, and proceed in the same manner.*

We have seen that a whole number may be converted into an equivalent fraction having any proposed denominator, by multiplying it by that denominator, and writing under it the same denominator. By this means, and by the rule just given, we are enabled to add a fraction to a whole number, or to subtract the one from the other, so as to express their sum or difference by a fraction. Let it be proposed to add $\frac{2}{3}$ to 4: we shall reduce 4 to a fraction whose denominator is 3, by multiplying it by 3, and subscribing the denominator 3; his fraction is $\frac{12}{3}$. This being added to $\frac{2}{3}$ gives $\frac{14}{3}$, which is the sum of 4 and $\frac{2}{3}$. If it be required to subtract $\frac{2}{3}$ from 4, we get the remainder $\frac{10}{3}$, which is the difference between 4 and $\frac{2}{3}$. In this way, a mixed number may always be converted into an improper frac-

tion. Thus, $7\frac{5}{6}$ may be converted into an improper fraction by adding 7 to $\frac{5}{6}$ by the above method. To do so we must multiply 7 by 6, and subscribe 6, and then add the fractions having the common denominator 6: the process would be as follows:—

$$7 + \frac{5}{6} = \frac{42}{6} + \frac{5}{6} = \frac{47}{6}.$$

(235.) The following is the

GENERAL RULE.

To convert a mixed number into an improper fraction — Multiply the integral part by the denominator of the fractional part, and to the product add the numerator of the fractional part. Taking this as numerator, subscribe the denominator of the fractional part.

(236.) Although the rule already given for reducing fractions to the same denominator is the only method which is universally applicable, yet it will happen in particular cases that more abridged processes may be found, arising from the peculiar relations of the numbers under consideration. Suppose, for example, that the fractions to be reduced to a common denominator were $\frac{3}{4}$ and $\frac{1}{2}$. If we followed the general rule in this case, we should obtain the equivalents $\frac{6}{8}$ and $\frac{4}{8}$; but a slight attention to the particular case proposed will show us that the denominator of the first fraction is twice that of the second, and that, consequently, if we multiply both terms of the second by 2, we shall obtain an equivalent fraction $\frac{2}{4}$, having the same denominator as the first. The equivalent fractions thus obtained are evidently in lower terms, and therefore more convenient than those which would result from the application of the general rule.

Again, if it were required to reduce the following fractions to a common denominator, —

$$\frac{2}{3} \quad \frac{3}{4} \quad \frac{5}{6} \quad \frac{7}{12} \quad \frac{2}{36},$$

we should observe that the denominator of the last is a multiple of each of the other denominators; and that, consequently, we can obtain a fraction equivalent to each of the others, having the same denominator as the

last, by multiplying both numerator and denominator by that number which would be found by dividing the last denominator by the denominator of the proposed fraction. Thus, since 36 is 3×12 , we shall obtain an equivalent for the first by multiplying both terms by 12. Again, since 36 is 4×9 , we shall obtain an equivalent for the second by multiplying both terms by 9. In the same manner, it will be perceived that we shall obtain equivalents for the third and fourth by multiplying their terms respectively by 6 and 3. The 5 equivalents with a common denominator thus obtained are $\frac{24}{36}$, $\frac{27}{36}$, $\frac{30}{36}$, $\frac{31}{36}$, $\frac{33}{36}$. If we applied the general rule in this case, the common denominator, instead of being 36, would be 31104, and the numerators would be proportionally great. The method here explained may be practised in every case where the greatest of the proposed denominators is a multiple of each of the others; but, although that may not happen, still it may happen that the greatest denominator multiplied by some smaller number may be a multiple of all the others; and, in such a case, the reduction to a common denominator may not only be considerably abridged, but the equivalent fractions, as in the above case, may be obtained in comparatively low terms.

Let the proposed fractions be $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{5}{6}$. The greatest denominator, 36, is not in this case a multiple of each of the others. Let us try, however, whether 36×2 is so. We find that $72 = 24 \times 3 = 18 \times 4 = 12 \times 6 = 8 \times 9 = 4 \times 18$. Hence it is apparent that we can obtain equivalent fractions with the denominator 72, by multiplying the terms of the first fraction by 18, those of the second by 9, of the third by 6, of the fourth by 4, of the fifth by 3, and of the sixth by 2. These fractions are the following:—

$$\frac{54}{72}, \frac{63}{72}, \frac{66}{72}, \frac{52}{72}, \frac{51}{72}, \frac{50}{72}.$$

It appears then, that the same fractions may be reduced to a great variety of different common denominators; and, indeed, this is sufficiently evident when

we consider that, after having found any set of equivalent fractions with the same denominator, these may be infinitely varied by multiplying all their terms successively by different numbers. We may by this means obtain an unlimited number of sets of equivalent fractions, the common denominator of each set being a multiple of the original common denominator.

(237.) Upon the general principle of convenience on which we prefer, in general, to use fractions in their least terms, so, in the present instance, whenever it is necessary to convert any set of fractions with different denominators into another equivalent set having a common denominator, we should select among all the possible common denominators that which is the least. This may be easily done.

Let us suppose equivalent fractions, with a common denominator, to be found by the rule already given. We are required to ascertain whether the common denominator thus obtained is the least possible; and, if not, to find what is the least possible. To accomplish this, find the greatest common measure of the several numerators of the equivalent fractions, and their common denominator. This common measure will divide both numerator and denominator of each of the fractions, and will convert them into another equivalent set, which will still have a common denominator. But since the divisor thus used is the *greatest* common divisor, the set of equivalent fractions obtained will be in the least terms which are consistent with having the same denominator.

To apply this rule, let it be required to reduce the fractions $\frac{1}{3}$, $\frac{5}{6}$, and $\frac{7}{8}$ to equivalent fractions having the least possible common denominator. By the general rule (232.) we should obtain the following results:—

$$\begin{array}{ccc} \frac{1 \times 6 \times 8}{3 \times 6 \times 8} & \frac{5 \times 1 \times 8}{6 \times 1 \times 8} & \frac{7 \times 4 \times 6}{8 \times 1 \times 6} \\ \text{or,} & & \\ \frac{1}{1} \frac{16}{48} & \frac{5}{1} \frac{60}{48} & \frac{7}{1} \frac{60}{48}. \end{array}$$

By the rule for finding the greatest common measure,
 37 4.

we find that the greatest common measure of 144, 160, 168, and 192 is 8. Dividing the numerator and denominator of each fraction by 8, we get the following equivalent fractions : —

$$\frac{18}{24} \quad \frac{20}{24} \quad \frac{21}{24}.$$

If the several numerators and common denominator of the first set of equivalent fractions obtained by the rule (232.) have no common measure greater than unity, then their common denominator is the least possible. This is evident, since the only way they could be reduced to lower terms, preserving the same denominator, would be by dividing their several numerators and their common denominator by the same number ; and that number must, therefore, be a common measure.

(238.) Although the least possible common denominator may always be found by the above rule, yet, in practice, we may sometimes abridge the process in the following manner : — If there be but two fractions given, find, in the first instance, the greatest common measure of their denominators. If the denominators be not great, this number will frequently be found by mere inspection. Multiply the two denominators together, and divide their product by their greatest common measure. The quotient will always be a multiple of each of the denominators, or a *common multiple* of them. Each of the fractions may be reduced to an equivalent one having this multiple as the denominator. The two fractions thus obtained will have the least possible common denominator. This process will be easily understood by means of an example.

Let the proposed fractions be $\frac{3}{4}$ and $\frac{5}{6}$: the greatest common measure of their denominators is 2. Now, if we multiply the denominators together, and divide the product by 2, we must necessarily obtain the same result as if we first divided either of the denominators by 2, and then multiplied the quotient by the other. It is evident that the same arithmetical operations are performed in each case, only in a different order. If

we divide 4 by 2, and multiply the quotient by 6, we shall obtain 12, which is a multiple of 6: or if we divide 6 by 2, and multiply the quotient by 4, we shall obtain a multiple of 4, being the same number, 12. Thus, in each case the result will be the same, and will be at once a multiple of each denominator. But we should obtain precisely the same result if we performed the same operations in another way, viz. by multiplying the denominators together, and dividing the product by 2.

To find two equivalent fractions having the common denominator 12, it is only necessary to find how often each denominator is contained in 12. The first being contained in 12 three times, we multiply both terms of the first fraction by 3, by which means it is converted into the equivalent fraction $\frac{9}{12}$. The denominator of the second fraction being contained in 12 twice, we multiply both terms of it by 2, by which means it is converted into the equivalent fraction $\frac{6}{12}$.

(239.) If it be required to reduce 3 or more fractions to the least possible common denominator, we may extend the application of this method without difficulty. Find the least common multiple of their common denominator and the denominator of the third fraction; next find the least common multiple of this and the denominator of the fourth, and so on. The common multiple last found will be the least common denominator to which the fractions can be reduced. They may be reduced to this denominator by finding how often the denominator of each fraction is contained in the proposed common denominator, and then multiplying both terms of the fraction by that number.

Let it be proposed to find the least common denominator to which the fractions $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{8}$, and $\frac{5}{9}$ may be reduced. First, find the greatest common measure of the first 2 denominators; in this case that common measure is 1: the least common multiple of these two is, therefore, 12 (238.). Next find the greatest common measure of 12 and 8, which is 4. Divide

12 by 4, and multiply the quotient by 8: the result is 24, which is the least common denominator of the first 3 fractions. Next find the greatest common measure of 24 and 9, which is 3. Divide 24 by 3, and multiply the quotient by 9: the result is 72, which is the least possible common denominator of all the proposed fractions.

To find the equivalents, find how often 3 is contained in 72: it is contained 24 times. Multiply both terms of the first fraction by 24, and the result is $\frac{38}{72}$. In like manner, find how often 4 is contained in 72: it is contained in 72 eighteen times. Multiply both terms of the second fraction by 18: the result is $\frac{18}{72}$. In like manner, 8 being contained 9 times in 72, multiply the terms of the third fraction by 9: and 9 being contained 8 times in 72, multiply the terms of the fourth fraction by 8. The series of equivalent fractions will then be $\frac{38}{72}$, $\frac{18}{72}$, $\frac{67}{72}$, $\frac{49}{72}$; and these are the equivalent fractions which have the least possible common denominator.

(240.) It is sometimes required to convert a given fraction into an equivalent, having some proposed number as its denominator, the given fraction being reduced to its least terms. It is only possible to effect this when the denominator proposed is an exact multiple of the denominator of the fraction. If it be so, multiply both terms of the proposed fraction by the quotient found by dividing the proposed denominator by the denominator of the fraction. For example: let it be proposed to convert $\frac{3}{4}$ into an equivalent fraction having 16 as a denominator: 4 divides 16 four times; therefore, multiply both terms of $\frac{3}{4}$ by 4, and we shall obtain $\frac{12}{16}$, which is the equivalent sought. Had it been required to convert $\frac{3}{4}$ into an equivalent fraction, having 18 for its denominator, the solution of the problem would be impossible; since there is no whole number by which the denominator of $\frac{3}{4}$ could be multiplied which would produce 18

CHAP. II.

THE MULTIPLICATION AND DIVISION OF FRACTIONS.

(241.) IN the first book we have contemplated multiplication as an abridged method of addition, by which we are enabled to determine, by a short process, what number would be obtained by the addition of the same number repeated any proposed number of times. The extension which has been given to the language and notation of number in the last chapter, renders it necessary that we should enquire what effect the same extension will produce upon the operations of multiplication as already described.

If we are required to determine the product of 12 multiplied by 8, the thing to be ascertained is the sum which would be obtained by the addition of 12 eight times repeated. Again: if we are required to determine the product of 12 multiplied by 9, it is necessary to find what sum would be obtained by 12 nine times repeated. In fact, the multiplicand must in this case be repeated in the addition as many times as there are units in the multiplier. Now, suppose that the multiplier, instead of being either 8 or 9, were $8\frac{3}{4}$, shall we repeat the multiplicand eight times or oftener? How many units are contained in $8\frac{3}{4}$? More than 8 units are contained in it, but less than 9: the multiplicand ought therefore to be repeated more than 8 times, but less than 9 times. The multiplier contains only a part of the ninth unit, therefore the multiplicand must be repeated only a part of the ninth time; that is to say, the whole multiplicand must not be repeated the ninth time, but only as much of it as there is of the ninth unit contained in the multiplier. Now, if the

ninth unit be imagined to be divided into four equal parts, three of these are taken in the multiplier. We must, therefore, to find the product required, first repeat the multiplicand 8 times, and then, instead of repeating the whole multiplicand the ninth time, we must divide it into 4 equal parts, and repeat only 3 of these. The total process by addition would then be as follows : —

$$\begin{array}{r}
 12 \\
 12 \\
 12 \\
 12 \\
 12 \\
 12 \\
 12 \\
 12 \\
 12 \\
 3 \\
 3 \\
 3 \\
 \hline
 105
 \end{array}$$

The multiplicand being divided into 4 equal parts, each of these parts is 3: after repeating the multiplicand 8 times, we therefore repeat its fourth part, 3, three times. Adding the whole, we obtain 105, which is the product sought.

So much of this process as consists in multiplying 12 by 8 is nothing more than the ordinary process of multiplication, as explained in our first book. But the analogous process of multiplying 12 by $\frac{3}{4}$ is an extension of multiplication to the new species of multiplier introduced to our notice in the last chapter. If we wish to multiply 12 by 3, we have only to repeat it three times, and add: if we wish to multiply 12 by $\frac{3}{4}$, we have only to repeat it *three fourths of a time*, or, what is the same, we have only to repeat its fourth part 3 times, and add; the product would be 9.

The constant habit which persons are given, of considering multiplication only as applied to integers, causes the fact of finding a product less than the multi-

plicand to have a startling effect. That the operation of multiplication should diminish and not increase that to which it is applied, is a matter which at first view seems paradoxical. It is hoped, however, that the above illustration will convince any one who gives the necessary attention to it that there is nothing contradictory or paradoxical in the matter. The word "multiplication" has received an extended meaning, which is perfectly consistent and analogous with its more popular acceptation, as applied exclusively to whole numbers. Without such an extension, the theory and practice of fractions would fall into inextricable complexity and confusion; and, indeed, it would be difficult to frame a system of language by which those arithmetical operations could be expressed in which mixed numbers are engaged. We should have one nomenclature for the operations performed on the integral parts of them, and another for the like operations performed on the fractional parts. Such a complicated phraseology would be equally perplexing and absurd. As, however, minds habituated to the consideration of the arithmetic of whole numbers, and of whole numbers only, commonly find some difficulty in this extension of the meaning of the term multiplication, we shall further illustrate the perfect consistency of the application of the term to fractions with its sense as applied to whole numbers.

Let us, as before, suppose that we are required to multiply a number by $\frac{3}{4}$; and let us take the terms multiplication and division in the sense in which they are usually understood when applied to whole numbers, viz. multiplication, as an operation by which something is increased in a certain proportion; and division, an operation by which it is diminished in a certain proportion.

We have already explained that $\frac{3}{4}$ may be considered as expressing the fourth part of 3. Being required to multiply by $\frac{3}{4}$, we are then required to multiply by the fourth part of 3. Let us suppose that we proceed by first multiplying the number proposed by 3. What-

ever the product may be which is obtained by this process, it is evidently greater than that which we seek; because we have multiplied by 3, instead of multiplying by the fourth part of 3. But it is further evident that it is not only greater than the product required, but greater, in exactly the same proportion, as 3 is greater than its fourth part: the product found is, therefore, 4 times too great. To diminish it to its just magnitude, it will, therefore, be necessary to divide it by 4: the quotient of such division will be the true product sought. Apply this to the case in which the multiplicand is 12: we are required to multiply 12 by the fourth part of 3: we multiply it by 3, and obtain 36; but having multiplied 12 by a number four times too great, the product 36 is four times the product which we seek. We therefore divide 36 by 4, and obtain 9, which must be the product sought: when 12 is multiplied by $\frac{3}{4}$, the product then is 9.

Under this point of view, the multiplication of a number by a fraction is a two fold operation; a multiplication by its numerator, and a division by its denominator. We must not, however, infer that the multiplication by a fraction, in its own essential nature, is a more complex process than the multiplication by a whole number; or, to speak more strictly, we must not conclude that the product bears to its factors a different relation when one of those factors is a fraction, from the relation it has to them when both are integers. The greater complexity is merely in the method adopted for practising the operation: the arithmetical relation of the numbers engaged in the question is, in both cases, precisely the same.

There is one circumstance more, which offers a forcible proof of the consistency of the extension of the term multiplication here contemplated. Taking the example of the multiplication of 12 by $\frac{3}{4}$, it will not be doubted that whether we multiply 12 by $\frac{3}{4}$, or $\frac{3}{4}$ by 12, we must needs obtain the same product. Now let us take the latter view of the question, and let us suppose

that we choose to perform the operation by multiplying $\frac{3}{4}$ by 12 : we are then to repeat $\frac{3}{4}$ twelve times, and add. Since all the fractions to be added in this case have the same denominator, 4, we shall add them by merely adding their numerators (234.), retaining the same denominators: 3, twelve times repeated, must therefore be added ; and the result is 36, the product sought being $3\frac{3}{4}$. But, by what has been already proved (225.), $3\frac{3}{4} = 9$: the product, then, of $\frac{3}{4}$ multiplied by 12 is 9, which is the same as the product already found by other reasoning when 12 was multiplied by $\frac{3}{4}$. If, then, it be admitted that $\frac{3}{4}$ ought to be multiplied by 12, by repeating $\frac{3}{4}$ twelve times, and adding, we must needs also admit that 12 should be multiplied by $\frac{3}{4}$, by dividing it into four equal parts, and repeating one of these parts three times, the result being the same in both cases. If this was not conceded, we should be forced into the absurdity of maintaining that 12 multiplied by $\frac{3}{4}$ is a different number, and the operation a different operation from $\frac{3}{4}$ multiplied by 12.

(242.) From all that has been above stated, we may infer the following

RULE.

To multiply any number by a fraction, we should multiply that number by the numerator of the multiplier, and divide the result by the denominator of the multiplier. The quotient will be the product sought.

(243.) A fraction may be multiplied by a whole number, by either of two methods ; one of which is always practicable, and is therefore preferable as a general rule, and the other more simple when it is practicable, and therefore advantageous as an occasional rule. By what has been proved above, it appears that a fraction may always be multiplied by a whole number, by multiplying its numerator by that number, preserving the denominator. This rule is perfectly general. When it happens, however, that the denominator is divisible

exactly by the whole number, then the fraction may be multiplied by the whole number, by dividing its denominator by that number (210.).

(244.) The multiplication of any number by a proper fraction will diminish it in the proportion of the numerator of the multiplier to its denominator; and the multiplication by an improper fraction will increase it, in the proportion of the numerator of the multiplier to its denominator. This will be easily perceived by considering attentively what has been already proved. If a number be multiplied by $\frac{4}{5}$, the product will be four fifths of the multiplicand (241.): consequently, the product will be less than the multiplicand in the proportion of 4 to 5. If a number be multiplied by $\frac{5}{4}$, the product will be five fourths of the multiplicand, and consequently will be greater than the multiplicand in the proportion of 5 to 4.

When a number is multiplied by a fraction, it is submitted successively, as we have observed, to the two operations of multiplication by the numerator, and division by the denominator. It is immaterial, in so far as regards the result, in which order these operations are performed; but it is frequently more convenient, first, to divide the multiplicand by the denominator, and then to multiply the quotient by the numerator. This method is always preferable when the multiplicand, being a whole number, is a multiple of the denominator. In the example above given, we are required to multiply 12 by $\frac{3}{4}$. The most simple method of proceeding is first to divide 12 by 4, and then multiply the quotient by 3. In this case we shall have small numbers to deal with, whereas in multiplying by 3, and dividing by 4, we shall have higher numbers. In the present instance, the numbers concerned in the question being small, this advantage is not so apparent; but in questions involving high numbers it is of some importance. Let us suppose that we are required to multiply 376625 by $\frac{37}{125}$. If we first multiplied by 37, and then divided by 125, the process would be of considerable length.

but by dividing^d the multiplicand first by 125, we get the quotient 3013, which multiplied by 37 gives the true quotient, 111481.³

It is, perhaps, scarcely necessary to observe, that in multiplication, as in most other operations in which fractions are engaged, it is convenient, in the first instance, to reduce the fractions to their least terms, in order to avoid the introduction of numbers unnecessarily large.

(245.) When the multiplicand is a whole number, and the multiplier a fraction, the product will always be an integer, when the multiplicand is a multiple of the denominator of the multiplier. This will be evident from what has been just proved; for if we divide the multiplicand by the denominator of the multiplier, we shall obtain a whole number for the quotient, and this whole number, being multiplied by the numerator of the multiplier, will give a whole number for the product. But if the multiplicand be not an exact multiple of the denominator of the multiplier, the latter being reduced to its least terms, then the division of the multiplicand by the denominator of the multiplier will give a quotient which is either a fraction or a mixed number; and this being multiplied by the numerator, it will still be either a fraction or a mixed number.

(246.) From what has been explained, it is easy to discover methods by which one fraction may be multiplied by another. It has been proved (242.), that when *any number* is required to be multiplied by a fraction, we have only to multiply that number by the numerator of the fraction, and to divide the result by its denominator. Let the multiplicand, then, be a fraction we must first multiply it by the numerator of the multiplier; that numerator being a whole number, and the multiplicand being a fraction, we must multiply the numerator of the multiplicand by that whole number, preserving the denominator (214.). It is next necessary to divide this result by the denominator of the multiplier (242.). Now, it has been proved that a

fraction is divided by a whole number, by multiplying its denominator by that whole number (215.). We must, therefore, in the present case, multiply the denominator of the result we have just obtained, by the denominator of the multiplier: the fraction which we shall thus find will be the product sought.

Let us apply this reasoning to an example.—Let the multiplicand be $\frac{3}{5}$, and the multiplier $\frac{2}{7}$: we must first multiply $\frac{3}{5}$ by 3, and then divide the result by 7 (242.). To multiply $\frac{3}{5}$ by 3, we multiply 4 by 3, preserving the denominator (214.), and obtain $\frac{12}{5}$: this must be divided by 7; but this division may be effected by multiplying its denominator by 7 (215.), and we shall thus obtain $\frac{12}{35}$, which is therefore the product $\frac{3}{5} \times \frac{2}{7}$.

(247.) Hence we may infer the following

RULE.

To find the product of two fractions multiply the numerators for a numerator, and the denominators for a denominator.

Whatever may be the terms in which the fractions are expressed, and whatever may be their mutual relation, this rule is always applicable, and therefore perfectly general. In particular instances, however, the multiplication may be performed more simply, and the product obtained in lower terms, by other methods, which are founded on the principles explained (243.).

(248.) Since the multiplicand may be multiplied by the numerator of the multiplier, either by multiplying its numerator, or dividing its denominator, we may adopt the latter method with advantage, whenever the denominator of the multiplicand happens to be an exact multiple of the numerator of the multiplier. Suppose, for example, the multiplicand is $\frac{3}{8}$, and the multiplier $\frac{2}{5}$. In this case, instead of following the general rule, we shall multiply $\frac{3}{8}$ by 2, by dividing its denominator by 2; the result is $\frac{3}{4}$: this being divided by 5, by

multiplying its denominator by 5, we obtain $\frac{3}{20}$, the product. Had we proceeded by the general rule we should have obtained the product in the terms $\frac{6}{40}$, which should be afterwards reduced to its least terms, $\frac{3}{20}$.

Since there is no real distinction between the multiplicand and multiplier, we may therefore infer, generally, that when the denominator of one fraction is an exact multiple of the numerator of the other, instead of multiplying the numerators, we may divide the denominator of the one fraction by the numerator of the other, and multiply the quotient by the other denominator.

(249.) If the denominator of one fraction happen to be *the same number* as the numerator of the other, then the product may be obtained without either multiplication or division, by merely retaining the numerator and denominator, which are dissimilar. Let the fractions to be multiplied be $\frac{2}{5}$ and $\frac{3}{4}$. If the latter be multiplied by 4, we have only to omit its denominator: the result being 3, must be divided by 5, which gives the fraction $\frac{3}{5}$ as the true product. If we apply the general rule in this case, we should first obtain $\frac{6}{20}$, which reduced to its lowest terms would be $\frac{3}{5}$.

(250.) If it happen that the numerator of one of the fractions is an exact multiple of the denominator of the other, the process may also be abridged. Let the fractions to be multiplied be $\frac{3}{4}$ and $\frac{8}{7}$. It is necessary to multiply the latter by 3, and to divide it by 4. Now, it may be divided by 4, by dividing its numerator by 4; the result would be $\frac{2}{7}$: this multiplied by 3 will give $\frac{6}{7}$, the complete product. Hence, when the numerator of one fraction is a multiple of the denominator of the other, divide the numerator of the former by the denominator of the latter, and multiply the quotient by the remaining numerator, preserving the remaining denominator.

(251.) It may so happen that the numerator of one factor may be a multiple of the denominator of the other, and also its denominator a multiple of the nu-

merator of the other. In such case the process of multiplication may be abridged.

Let the fractions to be multiplied be $\frac{3}{5}$ and $\frac{2}{3}\frac{5}{6}$. It is necessary to multiply the latter by 3, and to divide the result by 5. We may multiply it by 3, by dividing its denominator by 3 (243.): the result will be $\frac{2}{1}\frac{5}{2}$. This must be divided by 5 (242.), which may be done by dividing its numerator by 5 (214.): the result will be $\frac{2}{1}\frac{5}{2}$, which is the product sought. Had we proceeded in this case by the general rule, we should obtain the product under the form $\frac{7}{180}$, which, reduced to its least terms, would be $\frac{2}{1}\frac{5}{2}$.

(252.) A similar means of abridgment may be adopted if the numerator of each factor be a multiple of the denominator of the other. Let it be required to multiply $\frac{3}{5}$ by $\frac{2}{3}$: we shall first divide $\frac{3}{5}$ by 3 (242.), the result will be $\frac{1}{5}$. This must be multiplied by 25; but we may multiply 25 by $\frac{1}{5}$, by first dividing it by 5, and then multiplying by 12 (242.): the result will be 60. When the numerators, therefore, are multiples of their alternate denominators, divide each by the alternate denominator, and multiply the quotients together. The product will in this case be always a whole number.

It would be equally tedious and unnecessary to explain the details of the various expedients which may be adopted for the abridgment of the multiplication of fractions in cases where the factors have peculiar numerical relations. It will be sufficient to observe that all such methods depend immediately on the principles proved in (214.) (215.).

(253.) If it be required to multiply a mixed number by a whole number, the most expeditious method is, first, to multiply the fractional part by the whole number, converting the product, if it be an improper fraction, into a mixed number; and next to multiply the integral part by the whole number; and, finally, add the results. Thus, if we are required to multiply $7\frac{2}{3}$ by 5, we first multiply $\frac{2}{3}$ by 5, which gives $\frac{10}{3}$; this reduced to a mixed number (235.) gives $3\frac{1}{3}$. The

integral part, 7, being multiplied by 5, gives 35, which being added to $3\frac{1}{3}$ gives $38\frac{1}{3}$, the total product.

(254.) When the two factors are the same fraction, the product, as in the case of integers, is called the *square* or *second power* of the fraction; and the terms *cube* or *third power*, *fourth power*, &c. are applied to fractions in the same sense as they are applied to whole numbers. It appears, therefore, that any power of a fraction is found by taking the same powers of its numerator and denominator for the numerator and denominator of the required power of the fraction. This will be evident by applying the rule for the multiplication of fractions to the determination of the powers of any proposed fraction. Let the fraction be $\frac{2}{3}$, its powers will be found in the following manner:—

$$2d \text{ power} - \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2}{3 \times 3}$$

$$3d \text{ power} - \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2}{3 \times 3 \times 3}$$

$$4th \text{ power} - \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3}$$

$$5th \text{ power} - \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3 \times 3}$$

&c. &c. &c.

It is evident that the numerators are the same powers of 2, and the denominators the same powers of 3, as the fractions themselves are of $\frac{2}{3}$.

(255.) All that has been explained respecting the multiplication of fractions, will render the process of division of fractions more easily comprehended. The same anomalous and paradoxical circumstance noticed in multiplication reappears in division. According to the common popular notion of division, derived exclusively from the case in which the divisor is a whole number, the effect which is produced upon the dividend is to diminish its amount; and it is accordingly expected that the quotient in this operation will be less than the dividend. It requires, however, very little attention to perceive, that when once the notions and language of fractions are introduced, there is no reason why the quotient should be less than the

dividend. Let it be recollected that the quotient expresses the number of times the divisor is contained in the dividend. Now, if the divisor be a proper fraction, it will be contained in the dividend a greater number of times than 1 is contained in the dividend, because, in that case, the divisor will be smaller than 1. Since, then, 1 is contained in the dividend as often as there are units in it, a proper fraction will be contained in it a greater number of times than there are units in it; and, consequently, in such a case, the quotient must needs be greater than the dividend. Let us suppose, for example, that the divisor is $\frac{1}{4}$. This divisor will be contained 4 times in every unit of the dividend. If the dividend, then, be a whole number, the quotient will be another whole number 4 times as great. Let the dividend, for example, be 5 : in each of the 5 units which compose the dividend, the divisor is contained 4 times ; consequently it is contained in the whole dividend 20 times. The quotient is, therefore, 20, and is 4 times as great as the dividend. It will be easily understood from this example, that there is nothing inconsistent or absurd in the supposition that the process of division should give a quotient greater than the dividend.

(256.) But let us take the question under another point of view. Suppose* that a certain number is required to be divided by $\frac{3}{4}$, or, what is the same, by the fourth part of 3. If we divide it, in the first instance, by 3, we shall evidently get too small a quotient, because we have used too great a divisor : we have divided by 3 instead of its fourth part. The divisor, then, being 4 times too great, the quotient will be four times too small. In order to compensate for this error, we should multiply the quotient by 4, and the product will be the true quotient sought. By generalising this reasoning, it will be apparent, that when it is required to divide any number by a fraction, it is only necessary, first, to divide that number by the numerator of the fraction, and then to multiply the

result by its denominator. It may be observed, also, that, as the result must be the same in whatever order these operations are performed, we may invert their order, and first multiply by the denominator of the fraction, and then divide the result by its numerator.

(257.) This rule for division by a fraction might have also been discovered from the consideration that division is a process exactly the reverse of multiplication; that, in fact, the dividend being imagined to be the product of the divisor and another number, the effect of the division is to undo the previous multiplication by the divisor. Since, therefore, the multiplication by a fraction was performed by multiplying by the numerator and dividing by the denominator, the reverse process of division must naturally be inferred to be executed by dividing by the numerator and multiplying by the denominator.

The whole practice of the division of fractions follows as an easy and obvious consequence of the principle just proved.

(258.) If the dividend be a whole number, we should, *in general*, first multiply it by the denominator of the divisor, and then divide the product by the numerator of the divisor.

(259.) But if it happen that the dividend is a multiple of the numerator of the divisor, then we should first divide by the numerator of the divisor, and multiply the quotient by its denominator. Let the divisor be $\frac{4}{5}$ and the dividend 22: first multiply 22 by 5, and divide the product 110 by 4. The quotient, $27\frac{1}{2}$, is the quotient of 22 divided by $\frac{4}{5}$.

If the dividend had in this case been 24, we should first have divided it by 4, and then multiplied the quotient, 6, by 5. The product, 30, would then be the quotient of 24 divided by $\frac{4}{5}$.

(260.) If the dividend be a fraction, the method of effecting the division will easily follow from what has been already proved. We must first divide it by the numerator of the divisor (257.). This may be always

done by multiplying its denominator by that numerator. This done, we must next multiply the result by the denominator of the divisor. This may be always effected by multiplying its numerator by that denominator. For example, let the divisor be $\frac{3}{4}$ and the dividend $\frac{5}{7}$. To divide $\frac{5}{7}$ by $\frac{3}{4}$, we must multiply its denominator by 3: the result is $\frac{5}{21}$. This must be multiplied by 4, which is done by multiplying its numerator by 4: the result is $\frac{20}{21}$, which is the quotient sought. Hence we obtain the following general

RULE.

To divide one fraction by another, multiply the denominator of the dividend by the numerator of the divisor, and the numerator of the dividend by the denominator of the divisor.

(261.) When two fractions consist of the same terms, but in an inverted position, one is called the *reciprocal* of the other: thus $\frac{5}{4}$ is the *reciprocal* of $\frac{4}{5}$, $\frac{3}{7}$ the reciprocal of $\frac{7}{3}$, &c.

The term reciprocal, thus explained, being understood, the rule for the division of fractions may shortly be expressed thus: —

(262.) *Multiply the dividend by the reciprocal of the divisor.*

(263.) There are various expedients by which, in particular cases, the division of fractions may be facilitated, all depending on the same principles as those from which similar methods of abridgment were inferred in multiplication. Since the division of one fraction by another is performed by multiplying the dividend by one integer, and dividing it by another, and since there are two methods by which each of these operations may in certain cases be performed, we may frequently choose among these methods that which is most convenient. In particular cases, then, instead of multiplying the denominator of the dividend by the numerator of the divisor, we may produce the same effect by

dividing the numerator of the dividend by the numerator of the divisor. This, however, can only be done when the numerator of the dividend happens to be an exact multiple of the numerator of the divisor. Suppose it is required to divide $\frac{1}{5}$ by $\frac{2}{3}$; instead of multiplying 5 by 2, we should divide 4 by 2: the result would be $\frac{2}{3}$. We multiply the numerator of this by 3, and obtain $\frac{2}{3}$, which is the quotient sought.

(264.) Again, instead of multiplying the numerator of the dividend by the denominator of the divisor, we may produce the same effect by dividing the denominator of the dividend by the denominator of the divisor. Let it be required to divide $\frac{6}{6}$ by $\frac{2}{3}$. We should divide 6 by 3, and multiply the quotient by 2: the result would be $\frac{5}{1}$, the quotient sought.

(265.) If it should so happen that the numerator and denominator of the dividend should both be exact multiples of the numerator and denominator of the divisor, we may, by combining both the above methods, proceed by dividing the numerators for a numerator, and the denominators for a denominator. Let it be required to divide $\frac{6}{15}$ by $\frac{2}{3}$: we should, in this case, divide 6 by 2 and 15 by 3, and the result would be $\frac{3}{3}$, the true quotient.

If the divisor and dividend have the same denominator, the division may be performed by dividing their numerators. In this case the quotient, which would be obtained by the general rule, would be a fraction, both whose terms would be divisible by the common denominator; and, being divided by it, would be reduced to a fraction whose terms would be the numerators of the dividend and divisor. Let the divisor be $\frac{1}{5}$, and the dividend $\frac{3}{5}$. By the general rule, the quotient would be $\frac{3 \times 5}{4 \times 5}$. Dividing both numerator and denominator by 5, it will become $\frac{3}{4}$.

(266.) If a fraction be multiplied by its reciprocal, the product will always be 1; for in that case the numerator and denominator of the product will be the same: thus, $\frac{3}{4} \times \frac{4}{3} = \frac{12}{12} = 1$.

(267.) On the other hand, a fraction divided by its reciprocal will give a quotient which is the square of the dividend (254.); for in that case the numerator will be the square of the numerator of the dividend, and the denominator the square of the denominator of the dividend: thus $\frac{3}{4}$ divided by $\frac{4}{3}$ gives as a quotient $\frac{9}{16}$, which is the square of $\frac{3}{4}$.

(268.) When it is necessary to multiply one mixed number by another, or a mixed number by a fraction, the best general method of proceeding is, first to convert the mixed number into an equivalent fraction (235.), and then proceed by the rule for the multiplication of fractions. For example, let it be required to multiply $7\frac{2}{3}$ by $5\frac{7}{8}$. These being converted into equivalent fractions, they become $\frac{23}{3}$ and $\frac{47}{8}$, which, being multiplied (247.), give the product $1\frac{1081}{24}$. This, being an improper fraction, may be reduced to an equivalent mixed number (224.), by dividing the numerator by the denominator: it is thus reduced to $45\frac{1}{4}$.

(269.) When mixed numbers are required to be divided, either by each other or by fractions, they must, as in multiplication, be converted into equivalent fractions: the division may then be performed by the general rule for the division of fractions. For example, let it be proposed to divide $12\frac{3}{4}$ by $6\frac{1}{3}$: converting these into equivalent fractions, they become $\frac{51}{4}$ and $\frac{19}{3}$; these being divided by the rule (260.), we obtain the quotient $1\frac{53}{20}$, which, reduced to a mixed number (224.), becomes $1\frac{7}{20}$.

(270.) Most of the methods of verification or proof of the arithmetical operations in whole numbers will equally extend to fractions. Thus, multiplication may be verified by division, and division by multiplication. The product, when found, divided by the multiplier, ought to give the multiplicand; or divided by the multiplicand, ought to give the multiplier. In like manner, in division, the quotient, multiplied by the divisor, ought to give the dividend. Methods of proof may also be resorted to, which consist in effecting two

operations, which ought to have the same result if correct. Thus, if it be required to multiply two fractions, we know that the same product should be obtained if one of them be doubled and the other halved. After performing the multiplication by the rule, we may then double or halve the numerator of one fraction, and the denominator of the other, and then perform the multiplication again on the fractions thus altered. If the product be the same, the result will be verified.

To verify division, we may double the divisor and dividend, for in that case the quotient should remain the same (211.). If the result, therefore, of the division, when the divisor and dividend are so altered, be the same as before, the verification will be obtained.

(271.) In the practical operations of arithmetic, we have sometimes occasion to consider one fraction as the whole to which another is referred. The fraction contemplated is then called a *fraction of a fraction*, or a *compound fraction*. In such cases, it is generally necessary to express the fractional number by a simple fraction related to the original unit. Let us suppose that we are required to express $\frac{2}{3}$ of $\frac{5}{7}$ by a simple fraction referred to the original unit; or, in other words, to express the actual value of two thirds of five sevenths in a simple fraction.

Considering 5 sevenths as the whole of which we wish to take 2 thirds, we must first take its third part, and then multiply that by 2: in other words, we must divide $\frac{5}{7}$ by 3, and multiply the quotient by 2. Thus, of the two fractions concerned in the question, we must multiply the numerators for a numerator, and the denominators for a denominator: the third part of $\frac{5}{7}$ is $\frac{5}{21}$, and twice this is $\frac{10}{21}$. It is evident, then, that $\frac{2}{3}$ of $\frac{5}{7}$ is the same as the product of $\frac{2}{3}$ and $\frac{5}{7}$; and, in general, a fraction of a fraction is the same as the product of the two fractions (247.).

(272.) Compound fractions may be removed from the original unit by more than two steps; in other words, we may be required to find the fraction of a fraction of a

fraction, &c. Let it be required to find $\frac{2}{3}$ of $\frac{5}{7}$ of $\frac{9}{11}$: by what has been just proved, $\frac{2}{3}$ of $\frac{5}{7}$ is the product $\frac{10}{21}$ of these two fractions: we are, therefore, to find $\frac{10}{21}$ of $\frac{9}{11}$, and this is, in like manner, the product of these two, and is $\frac{90}{231}$; but 90 is the continued product of 2, 5, and 9, and 231 is the continued product of 3, 7, and 11. Hence a compound fraction of the third order is the continued product of the fractions of which it is composed.

By the same reasoning it will easily appear that a compound fraction of the fourth or any higher order is the continued product of its component fractions. To find $\frac{2}{3}$ of $\frac{5}{7}$ of $\frac{9}{11}$ of $\frac{1}{15}$, we must first find $\frac{2}{3}$ of $\frac{5}{7}$, which is the product of these fractions: we must next find the last fraction of $\frac{9}{11}$, which is the continued product of the first three component fractions; we must finally determine this last fraction of $\frac{1}{15}$, the result of which is the compound fraction required, and is evidently the continued product of the component fractions.

(273.) A whole number or a mixed number may be one of the component parts of a compound fraction. In such a case it is combined with the other parts by multiplication, in the same manner as if it were a fraction. If it be a whole number, it may be considered as a fraction having 1 as its denominator: if it be a mixed number, it may be reduced to an improper fraction.

CHAP. III.

DECIMALS.

(274.) THE decimal nomenclature of number, as explained in the first Chapter of our first Book, has a minor but no major limit: in descending, it stops at units; but in ascending, a regular system of names is contrived, which may be continued indefinitely; and there is, accordingly, no number so great that it may not be expressed by appropriate terms. But, on the other hand, no number less than 1 is capable of expression by the system of language there explained. In that stage of our progress, however, we limited our notions to that class of numbers which are multiples of the unit. Having extended our views, and increased the range of our notions of number, in the present Book, so as to imagine and reason about numbers less in magnitude than unity, it is natural to enquire whether the admirable nomenclature, which has been so universally and successfully adopted to express numbers in the ascending scale above unity, may not admit of such a modification as may render it equally unlimited in the descending scale below that term. We should thus consider unity as the centre of the numerical system, with an infinite ascending and descending series above and below it. It would evidently, also, contribute to the beauty and perfection of such a nomenclature; if it could be so contrived that the descending series below unity should harmonise in its nomenclature with the ascending series above it.

It requires but little attention to the nature of the decimal nomenclature contrived to express whole numbers, to discover a corresponding nomenclature in perfect keeping with it, by which numbers less than the unit may be expressed. As the successive or-

to the right of that place would express *tenths*, the next digit to the right would express *hundredths*, and the following one *thousandths*, and so on. The system of numeration thus extended is exhibited in the following example : —

9	7	8	6	5	3	2	4	7
<i>Ten Thousands.</i>	<i>Thousands.</i>	<i>Hundreds.</i>	<i>Tens.</i>	<i>Units.</i>	<i>Tenths.</i>	<i>Hundredths.</i>	<i>Thousandths.</i>	<i>Ten-Thousandths.</i>

It is necessary, however, here to observe, that so long as the line of number always terminated in descending at the units' place, the first digit on the right was always known to express the original units; and the orders of units expressed by the other digits were always known by the relative positions of these digits with respect to the first. By the extension now proposed, the series of places would be equally unlimited on the right and on the left, and there would, therefore, be no means of designating the units' place, with reference to which the value of all other places on both sides must be determined. To remedy this inconvenience, some means should be adopted by which the units' place would be marked. To write over it the word *units*, is the first method which would suggest itself; but it is evident that any other shorter mark would equally answer the purpose, provided its adoption were universal. It has been accordingly agreed by all modern nations, who have cultivated arithmetic to any considerable extent, to mark the units' place by a dot between it and the place of tenths. Thus, the above number, in which the values of the several digits are indicated by the terms denoting those values written above them, would be expressed with equal clearness, disencumbered of the words inscribed, in the following manner : —

97865·3247.

The dot* placed on the right of 5 is the means of expressing that 5 is the units' place. That being once understood, all the other places, both to the left and to the right, become known by their relative position with respect to the units' place; and after a little practice, the number may be read with as much facility as if the names of the units were written above the several digits.

(275.) The dot is called the *decimal point*; the numbers expressed by digits to the right of that point are called *decimal fractions*, or, shortly, *decimals*.

(276.) A number is said to have so many *decimal places*, as there are digits in it to the right of the decimal point. Thus, the number 86·40032 has five decimal places.

(277.) When the point does not appear in a number, the number is always understood to be a whole number, and the proper place for the point would be immediately after the last figure on the right.

A number which consists solely of decimal places, must always have the decimal point before the first figure on the left. Thus, ·2376 is such a number, and signifies 2 tenths, 3 hundredths, 7 thousandths, and 6 ten-thousandths. In this case the presence of the dot is necessary, because otherwise the number would be understood to be a whole number. To prevent the possibility of the dot escaping the eye in such cases, it is sometimes usual to place before it a nought, thus 0·2376; the nought indicating the absence of all significant digits from the integral places.

(278.) From what has been stated, it will be evident that the local value of every digit in a number will depend upon its place with respect to the decimal point. If, therefore, the decimal point be removed from one position to another, the local value of every digit will undergo a change, and this change will be the same for all the digits, since their distances from the decimal

* A comma is sometimes used instead of the dot, thus :—97865,3247.

point will be equally increased or decreased. Let us suppose that, in the following number,

376.531,

the point is transferred between the 7 and 6, thus, 37.6531. The 3, which before expressed *hundreds*, now expresses *tens*, being one place nearer to the point; the 7, which before expressed *tens*, now expresses *units*, being next the point; the 6, which before expressed *units*, now expresses the *tenths*, being removed from the left to the right of the point. In the same manner, each of the decimal places, being one place farther from the point, is diminished to a tenth of its former value. Thus the local values of all the digits are reduced to a tenth of their former values, and the number is, in fact, *divided by 10*. For like reasons, if we had removed the point one place to the right, the local values of the digits would be increased in a tenfold proportion, and the number would be *multiplied by 10*. It appears, then, that a number may be multiplied or divided by 10, merely by changing the position of the point one place to the right or one place to the left.

(279.) By pursuing this reasoning, it will appear that, to multiply a number by 100, we have only to move the point two places to the right; and that we may divide it by 100 by removing it two places to the left. In the same manner the number may be multiplied or divided by 1000, by removing the point three places to the right or to the left, and so on.

(280.) Since the point is always understood after the last figure of an integer, the annexing of ciphers to the right is equivalent to the removal of the point so many places to the right; and, therefore, the consistency of the above results with what has been already proved (64. et seq.) will be apparent.

(281.) When there are no places on the left of the point, the point may still be removed any number of places to the left, by interposing so many ciphers between it and the first digit of the decimal. Thus, if

we would remove the point in the number $\cdot 2376$, three places to the left, we have only to interpose three ciphers between the point and 2. Thus the number would be expressed $\cdot 0002376$, which is the former number divided by 1000.

(282.) It has been formerly observed, that ciphers placed on the *left* of a whole number produce no effect upon its value, because they have themselves no absolute values, and do not change the position of any significant digit with respect to the units' place. The same observation will apply with equal force to ciphers annexed to the *right* of a decimal. Such ciphers having themselves no absolute values, and the significant digits of the decimal holding the same place with respect to the point as before, they must retain their former values. Thus, if to the number $\cdot 2376$ we annex 3 ciphers, it will become $\cdot 2376000$. Each of the digits 2, 3, 7, and 6 will here have the same local value as before: the 2 will express *tenths*, the 3 *hundredths*, and so on; the ciphers annexed will express nothing. If, in the course of any arithmetical operation, therefore, we should obtain a decimal terminating in ciphers, such ciphers may be omitted. Also, if any arithmetical process should be facilitated by annexing ciphers to a decimal, such ciphers may be annexed, since the value of the decimal is not changed by their presence.

(283.) Since the decimal point is always understood to be placed on the right of the units' place in an integer, it will follow from what has been already proved (278.), that an integer may always be divided by 10 by introducing the decimal point between the units' and tens' place. On the same principle it may be divided by 100 by interposing the decimal point between the place of tens and hundreds, and by 1000 by interposing the decimal point between the place of hundreds and thousands; and, in general, it may be divided by a number consisting of 1 followed by any number of ciphers, by cutting off to the right as many places of decimals as there are ciphers in such divisor. For ex-

ample, the number 76453 divided by 10 would be 7645.3; divided by 100 it would be 764.53; divided by 1000 it would be 76.453. It appears, therefore, that the following numbers are equivalent:—

$$\begin{array}{ll} 7645.3 = \frac{76453}{10} & 764.53 = \frac{76453}{100} \\ 76.453 = \frac{76453}{1000} & 7.6453 = \frac{76453}{10000} \\ .76453 = \frac{76453}{100000} & .076453 = \frac{76453}{1000000} \end{array}$$

In general, therefore, a number consisting either of decimal places only, or of places partly integral and partly decimal, may be converted into an equivalent fraction by writing the number itself without any decimal places as numerator, and 1 followed by as many ciphers as there are decimal places as denominator.

(284.) It appears, from these observations, that any number containing decimal places may be considered under different points of view, and expressed in words, or in vulgar fractions, in different ways. According to the principles on which the nomenclature and notation of decimals have been explained, we should read such a decimal as 76.453 in the following manner:—76 units, 4 tenths, 5 hundredths, and 3 thousandths, or should express it in figures as follows:—

$$76 + \frac{4}{10} + \frac{5}{100} + \frac{3}{1000}.$$

In this point of view, the number is regarded as an integer, followed by a series of fractions having the denominators 10, 100, 1000, &c. But from what has been stated above, we may also consider the number to express a single fraction whose denominator is 1000, and under this point of view it would be expressed in words thus:—76453 *thousandths*, or in figures, $\frac{76453}{1000}$. We may also consider the integral part separated from the decimal, and the latter as a single fraction whose denominator is 1000, in which case it would be expressed in words thus:—76 and 453 *thousandths*; or in figures thus, $76 \frac{453}{1000}$.

(285.) Any common fraction having a denominator consisting of 1 followed by ciphers, may be immediately converted into a decimal by writing down the nume-

rator alone, and cutting off as many places to the right as there are ciphers in the denominator. This follows evidently from what has been just stated. For example, let the fraction be $\frac{76453}{10000}$; we may omit the denominator, and write it as a decimal, thus, 76·453.

If, in this case, there should be more ciphers in the denominator than there are places in the numerator, the decimal places must be supplied by ciphers placed on the left: thus, if the fraction be $\frac{76453}{1000000}$, the equivalent decimal will then be 0·0076453.

It appears, therefore, that every number, including decimal places, may immediately be converted into a vulgar fraction, and every vulgar fraction whose denominator is 1 followed by ciphers may be immediately converted into a decimal.

(286.) Having shown that decimals are always reducible to equivalent vulgar fractions, we shall be enabled to deduce from the established properties of the latter, methods by which the various elementary arithmetical operations may be performed on decimals.

(287.) When decimals have the same number of decimal places, their equivalent fractions have the same denominators; and since, by what has been already proved (282.), we may annex ciphers to the right of a decimal without changing its value, so as to increase the number of decimal places at pleasure, we may always by this means equalise the number of decimal places in several decimals, and thereby reduce their equivalent fractions to a common denominator.

For example, let the proposed decimals be 12·506, 0·34, 6·0356, 23·4. The greatest number of decimal places here is four: we shall reduce, therefore, all the decimals to four decimal places, by supplying the necessary number of ciphers in all that are deficient; the decimals will thus become 12·5060, 0·3400, 6·0356, 23·4000. All these are equivalent to fractions having the denominator 10000.

(288.) It will be remembered that vulgar fractions are added and subtracted by first reducing them to the

same denominator, adding or subtracting their numerators, and then subscribing their common denominator. (234.) This rule may be at once transferred to decimals:—"To add or subtract decimals, equalise their decimal places, add or subtract them as if they were whole numbers, and take in the result the same number of decimal places." The reason of this rule is obvious: by equalising the decimal places we reduce the equivalent fractions to the same denominator (287.); by adding or subtracting as whole numbers the decimals thus modified, we add or subtract the numerators of the equivalent fractions, and by taking in the result the same number of decimal places, we subscribe their common denominator. (284.) The rule is thus brought strictly under that established for the addition or subtraction of vulgar fractions.

Let it be required to add the following decimals, 32·4056, 245·379, 12·0476, 9·38, 459·2375. We shall equalise the decimal places in these by annexing one cipher to the second and two to the fourth: this being done, let the decimals be added as whole numbers; the result will be as follows:—

$$\begin{array}{r}
 32\cdot4056 \\
 245\cdot3790 \\
 12\cdot0476 \\
 9\cdot3800 \\
 459\cdot2375 \\
 \hline
 758\cdot4497
 \end{array}$$

We have here pointed off four places in the total obtained, the equivalent fraction thus having 10000 for its denominator, which is the same as the denominator of the several decimals which are added together.

In practice it is not necessary or usual to annex the ciphers to fill the deficient places: it will be sufficient so to range the numbers one under the other, that the decimal point of one number shall be immediately under that of the other. It is evident that, if this is attended to, the result of the operation will be the same as if the ciphers

were supplied: the above number would then stand thus: —

$$\begin{array}{r}
 32\cdot4056 \\
 245\cdot379 \\
 12\cdot0476 \\
 9\cdot38 \\
 459\cdot2375 \\
 \hline
 758\cdot4497
 \end{array}$$

Let it be required to subtract $23\cdot0784$ from $62\cdot09$: placing the numbers as above, the operation will stand thus: —

$$\begin{array}{r}
 62\ 09 \\
 23\cdot0784 \\
 \hline
 39\cdot0116
 \end{array}$$

The subtraction is here performed as if the deficient places above the 8 and 4 of the subtrahend were supplied by ciphers, and the number of decimal places in the remainder is the same as it would be in the minuend and subtrahend had the places been supplied. Thus the minuend and subtrahend are virtually reduced to the same denominator, and that denominator is preserved in the remainder.

(289.) We shall with equal facility derive the rule for the multiplication of decimals from that established for the multiplication of vulgar fractions. If the decimals be multiplied as whole numbers, we shall, in fact, multiply the numerators of the equivalent fractions; but to obtain the denominator of the product, it is necessary to multiply the denominators of the equivalent fractions: these denominators are two numbers expressed by 1, followed by as many ciphers as there are decimal places; and the product of two such numbers will be 1 followed by as many ciphers as there are in both denominators taken together: thus, if there be three decimal places in one of the numbers, and four in the other, the denominators of the equivalent fractions will be 1000 and 10000; the product of these will be 10000000: this being the denominator of the product,

it will follow that we must take in the product 7 decimal places ; that is, as many decimal places as are in the two numbers taken together. To multiply decimals, therefore, we have the following rule : — “ Multiply them as whole numbers, and take in the product as many decimal places as are in the multiplicand and multiplier taken together.”

Let it be required to multiply 35·407 by 12·54 ; the process will be as follows : —

$$\begin{array}{r}
 35\cdot407 \\
 12\cdot54 \\
 \hline
 141628 \\
 177035 \\
 70814 \\
 35407 \\
 \hline
 444\cdot00378
 \end{array}$$

In this example there are 3 decimal places in the multiplicand, and 2 in the multiplier : we, therefore, take 5 in the product.

The above rule for the multiplication of decimals may also be explained as follows : — If we remove the dot from the multiplicand in the above number, we multiply it by 1000 (279.), and by removing the dot from the multiplier, we multiply it by 100 ; the figures are thus converted into whole numbers, and their product found. But since one was previously multiplied by 1000 and the other by 100, the product will be 100000 times too great ; consequently, to reduce it to its true value, we must divide it by 100000 ; but this is done by taking in it five decimal places.

If one only of the numbers to be multiplied includes decimal places, we must then take in the product as many decimal places as it contains ; thus, if we are required to multiply 23 by 4·57, we multiply it by 457, considered as a whole number ; the product will be 100 times too great, and consequently will be reduced to its true value by taking in it two decimal places.

(290.) It may happen that, after the product has

been found, the total number of places it contains may be less than the decimal places in the two numbers multiplied. In this case, it will be necessary, in supplying the decimal point, to fill the deficient places on the left by ciphers.

Let it be required to multiply 0·03054 by 0·023 ; the process will be as follows : —

$$\begin{array}{r}
 3054 \\
 23 \\
 \hline
 9162 \\
 6108 \\
 \hline
 70242 \\
 \hline
 0\cdot00070242
 \end{array}$$

We have here, in the first instance, multiplied the decimals, considered as whole numbers ; but their product, 70242, had only five places, while the number of decimal places in the numbers to be multiplied amounted to eight : it was necessary to take eight decimal places in the product, and therefore three ciphers were interposed between the decimal point and the first figure of the product found.

(291.) It has been proved that two fractions may be divided one by the other when they have the same denominator, by merely dividing their numerators, omitting altogether their denominators. (265.) This principle renders the process for the division of decimals extremely simple. Their equivalent fractions may be always reduced to the same denominator by equalising their decimal places. The general rule, therefore, for the division of decimals is, to “ Equalise the decimal places, expunge the decimal points, and divide them as whole numbers ;” this being obviously equivalent to reducing them to the same denominator, and dividing their numerators.

Let it be required to divide 43·047 by 2·53698. The number of decimal places will be equalised by annexing two ciphers to the dividend: this being done, and

the decimal point removed, the process of division will be as follows :—

$$\begin{array}{r}
 253698 \) \ 4304700 \ (\ 16 \\
 \underline{253698} \\
 1767720 \\
 \underline{1522188} \\
 245532
 \end{array}$$

The integral part of the quotient is then 16, and the remainder is 245532 : this remainder being less than the divisor, the division can be carried no farther in whole numbers ; but the total quotient may be made up by adding the fraction $\frac{245532}{253698}$ to 16 (208.); the total quotient is then $16 \frac{245532}{253698}$.

It will be observed that the quotient found in this way is not itself a decimal, but is a mixed number, one part being a whole number, and the other a vulgar fraction. If it be required to exhibit the quotient as a decimal, it will be necessary to convert the vulgar fraction into an equivalent decimal fraction : the method of doing this we shall presently explain.

If, after equalising the decimal places and expunging the point, the divisor be greater than the dividend, the division, as whole numbers, cannot be effected ; and the quotient can only be expressed by a fraction whose numerator and denominator are the dividend and divisor thus changed. For example, suppose it is required to divide 0·13 by 4·7 ; equalising the decimal places and expunging the point, the numbers will become 13 and 470 ; the quotient will be expressed by the fraction $\frac{13}{470}$.

The quotient can only be expressed in decimals, after we have obtained a method of converting vulgar fractions into decimals.

(292.) It has been proved (175.) that any increase which takes place in the dividend produces a corresponding increase in the quotient, the divisor being supposed to remain the same : thus, if, preserving the divisor, we multiply the dividend by 100 or 1000, we

multiply the quotient also by 100 or 1000. If we would, in such a case, restore the quotient to that value which it would have had if no increase had taken place in the dividend, we must obviously, in such a case, divide it by 100 or 1000. This observation will lead us to an easy method of converting a vulgar fraction into an equivalent decimal.

Let it be proposed to reduce the fraction $\frac{13}{25}$ to an equivalent decimal. It has been shown that $\frac{13}{25}$ means the quotient which would be obtained by dividing 13 by 25. Now, if we multiply 13 by 100, and then perform the division by 25, we shall obtain a quotient 100 times greater than that which we should have had if we divided 13 by 25 without any previous change. Performing the division as above stated, we obtain the following result :—

$$\begin{array}{r} 25 \overline{) 1300} \quad (52 \\ \underline{125} \\ 50 \\ \underline{50} \\ 0 \end{array}$$

The quotient of 1300 divided by 25 is then 52 ; but this quotient is necessarily 100 times greater than the quotient of 13 by 25, since the dividend has been multiplied by 100 : if we divide 52 by 100 we shall therefore obtain the true quotient of 13 by 25 ; but 52 may be divided by 100, by placing the decimal point before the 5. (283.) The decimal 0.52 is, therefore, the true quotient of 13 by 25, and is equal to the fraction $\frac{13}{25}$.

If this process be examined, it will be perceived that, in order to reduce the fraction $\frac{1}{5}$ to an equivalent decimal, we have annexed two ciphers to the numerator, have then divided, as we should with integers, by the denominator, and have taken two decimal places in the quotient. By this means the numerator of the proposed fraction is converted into a number which is exactly divisible by the denominator, without a remainder. If we had annexed only one cipher, so as to convert the

numerator into 130, we should have had a remainder, so that the quotient would not be complete. If, on the other hand, we had annexed more than two ciphers, there would be no remainder after the division had proceeded as far as the second cipher; and the third and all the succeeding figures of the quotient would be ciphers: the equivalent decimal would still be obtained, but it would terminate in one or more useless ciphers.

Let us suppose that we had annexed 4 ciphers to the numerator; the quotient would then have been 5200: having multiplied the numerator by 10000, this quotient will be 10000 times greater than the true quotient of 13 divided by 25: to reduce it to its true value, we must, therefore, divide it by 10000, which will be done by taking in it 4 decimal places; the true quotient will then be 0.5200; but the two final ciphers here are insignificant and useless (282.), and the equivalent decimal, as before, is 0.52.

(293.) From this we may collect the following general rule for converting a vulgar fraction into an equivalent decimal.

Place the numerator as dividend, and the denominator as divisor, and annex a cipher to the former; divide the numerator, with the cipher so annexed, by the denominator, and write down, as in division, the first figure of the quotient; to the remainder, if there be any, annex another cipher, and divide as before to obtain the second figure of the quotient, and continue to annex a cipher to every remainder, writing down the successive figures of the quotient, as in division. When a remainder is obtained, which, with a cipher annexed, is exactly divisible by the denominator, the operation will be complete. Point off as many decimal places in the quotient as there were ciphers annexed, and the decimal thus obtained will be equivalent to the vulgar fraction.

It will be perceived in this process, that annexing the ciphers to the several remainders is equivalent to annexing them to the original numerator; so that, in fact, the numerator has been by this means multiplied by 1 fol-

lowed by as many ciphers as have been annexed ; and by taking the same number of decimal places in the quotient, the quotient is divided by that number. These two processes, as already explained, neutralise each other : the ciphers annexed to the dividend would render the quotient too great, in the proportion of a number expressed by 1 followed by the ciphers to unity. By taking the same number of decimal places in the quotient, it is diminished in exactly the same proportion, and therefore restored to its true value.

(294.) If the numerator of the fraction be greater than the denominator, the first step in the division will be effected without annexing ciphers. In that case, when the division is complete, the quotient will contain a greater number of places than the number of ciphers which have been annexed, and the equivalent decimal will, therefore, have one or more places on the left of the point. This is a circumstance which might be easily anticipated ; for if the numerator of the fraction be greater than its denominator, the fraction must be greater than 1, and therefore a part of its decimal expression must consist of whole numbers.

Let the fraction be $\frac{197}{16}$; the process for converting this into a decimal will be as follows :—

$$\begin{array}{r}
 16 \overline{) 197} \quad (12 \cdot 3125 \\
 \underline{16} \\
 37 \\
 \underline{32} \\
 50 \\
 \underline{48} \\
 20 \\
 \underline{16} \\
 40 \\
 \underline{32} \\
 80 \\
 \underline{80} \\
 \dots
 \end{array}$$

We have here taken four decimal places in the quotient, having annexed a cipher to each of the last four remainders.

(295.) It must be evident that every case in which the division of one whole number by another is incomplete, in consequence of the last remainder being less than the divisor, may by these means be continued so as to obtain the remainder of the quotient in decimals. It is only necessary to annex a cipher to the last remainder, and continue the division in the same manner, annexing ciphers until a remainder is found which, with a cipher annexed, is exactly divisible by the divisor. Let as many decimal places be then taken in the quotient as there were ciphers annexed.

(296.) It sometimes happens that, in converting a fraction into an equivalent decimal, it is necessary to annex two or more ciphers to the numerator, before any figure of the quotient is obtained: the rule already given will still apply to this case; but as the number of places in the quotient will, under these circumstances, be less than the number of ciphers annexed, it will be necessary to place ciphers to the left of the quotient, in order to make up the necessary number of decimal places.

Let it be proposed to convert the fraction $\frac{3}{125}$ into an equivalent decimal; the process will be as follows:—

$$\begin{array}{r} 125 \overline{) 300} \quad (.024 \\ \underline{250} \\ 500 \\ \underline{500} \\ \dots \end{array}$$

The quotient, in this case, being 24, and having only two places, while three ciphers have been annexed, it is necessary to place a cipher to the left of the 2, in order to make up three decimal places, which it is necessary to take in the quotient.

(297.) In any number, whether integral or decimal,

the local value of a unit, occupying any place, is greater than the total value of the digits to the right of that unit, to whatever number of places such digits may extend. This, which is evidently true with whole numbers, is not less apparent in decimals. It is evident that a unit in the place of thousands must have a greater value than the total amount of any digits which can fill the inferior places of hundreds, tens, and units: the greatest digits which can occupy these places are nines, and if they were all filled by nines, their total amount, 999, would still be 1 less than the value of a single unit in the thousands' place. The same reasoning will hold good for any other place in the line of integers.

It is the same with decimals. In the decimal $0\cdot54376$ a unit in the second place from the point is greater than the aggregate value of all the digits which succeed it; for if all those digits were nines, they would have the greatest value which by possibility could be conferred on them, and yet they would still be less than a single unit in the second place from the point, as may be easily proved. The total value of the nines filling the places just mentioned would be $0\cdot00999$: now if to this number we add $0\cdot00001$, we shall obtain the number $0\cdot01000$; but this number is, in fact, a unit in the second place from the point, and since it is obtained by *adding* a certain number to $0\cdot00999$, it must be a greater number than the latter. The same reasoning will apply in every case, and we therefore infer that a single unit occupying any decimal place, is of greater value than the total amount of all the places to the right of it, however numerous those places may be. Thus, in the number $0\cdot54376$, a single unit of the 5, that is, a tenth, is greater than the value of all the succeeding figures. In the same manner the total value of $0\cdot00376$ is less than a hundredth, and the value of $0\cdot000376$ is less than a thousandth, and so on. We shall presently see the importance of this conclusion.

(298.) In the process for converting a fraction into a decimal, or for continuing the operation of division where a remainder is found less than the divisor, by annexing decimal places to the quotient, we have supposed that by continuing the operation a remainder will at length be found, which, with a cipher annexed, will be exactly divisible by the divisor. This, however, is frequently not the case, and it will happen that the division may be continued without end, remainders continually arising, none of which are divisible exactly by the divisor. In such a case it is impossible to express the exact quotient by decimals; nevertheless we may obtain a number expressed in decimals differing from the exact quotient by as small a quantity as may be desired. This will be easily understood when it is remembered that a single unit in any decimal place is of greater value than the total amount of all the decimal places which can follow it, however numerous these may be. Suppose that the process of division be continued by annexing ciphers until ten ciphers have been annexed; it would be necessary then to take ten decimal places in the quotient: had the operation been continued, the remainder of the quotient would be expressed by digits occupying decimal places to the right of the tenth place. The total value of such digits, however numerous they might be, could never amount to the value of a single unit in the tenth place of decimals; the deficient part of the quotient would therefore be less than the $\frac{1}{10000000000}$ th part of the original unit. To whatever extent, therefore, the division be carried by annexing ciphers, the remainder of the quotient will always be a proportionally small part of the unit; and as there is no limit to the extent to which we may carry the operation, so there is no limit to our approximation to the true quotient.

Let it be required to divide 294 by 7.356 , and to obtain a decimal differing from the true quotient by less than the $\frac{1}{10000}$ th part of the unit; the process is as follows: —

$$\begin{array}{r} 7356 \) \ 294000 \ (\ 39 \cdot 9673 \\ \underline{22068} \end{array}$$

$$\begin{array}{r} 73320 \\ 66204 \\ \hline 71160 \\ 66204 \\ \hline 49560 \\ 44136 \\ \hline 54240 \\ 51492 \\ \hline 27480 \\ 22068 \\ \hline 5412 \end{array}$$

We have first equalised the decimal places by annexing three ciphers to the dividend ; the decimal points are then removed, and the numbers are treated as whole numbers (291.). The first figures of the quotient are obtained without annexing any more ciphers to the dividend, and there are subsequently ciphers annexed to four remainders : four decimal places are therefore taken in the quotient (293.). But there is still a final remainder, and therefore, if the process were continued, the quotient would contain decimal places to the right of its last figure. The total value of these decimal places, however, would be less than the value of a single unit in the last place of the quotient : that value being the 10,000th part of the unit, it follows that the quotient above found is less than the true quotient by a number less than the 10,000th part of the unit.

BOOK III.

COMPLEX NUMBERS.

CHAPTER I.

OF COMPLEX NUMBERS IN GENERAL. — THEIR REDUCTION AND SIMPLIFICATION.

(299.) **SIMPLE** numbers are those which are formed by the aggregation of the same primary or original units, and to such the investigations in the preceding Book have been confined ; for although the units expressed by digits occupying different places possess different values, yet the relations which these values bear to the primary or original unit are always explicitly denoted by the position of the digit. We shall now direct our attention to another class of numbers, into which units of different magnitudes enter, and which are thence called *complex* or *compound numbers*.

Complex numbers owe their origin to the inconvenience and difficulty found in the ordinary affairs of life in the use of very high numbers. When quantities are to be expressed which would require high numbers, the difficulty is therefore avoided by adopting a large unit ; but if such a unit alone were adopted, a similar inconvenience would arise when very small quantities of the same kind are expressed, for in that case very complex and inconvenient fractions would be unavoidable. Several units of different magnitudes are therefore employed ; and the quantity, instead of being expressed by one number consisting of many places of figures, is expressed by several numbers, each having different units.

Let us suppose that a penny were adopted as the unit of numbers expressing money ; a pound sterling would then be expressed by the number 240, ten pounds by

2400, and so on. These, however, being sums which it is necessary frequently to express, the use of such high numbers would be attended with manifest inconvenience. If, on the other hand, a pound were taken as the pecuniary unit, such a sum as a penny, which it is likewise necessary frequently to express, could only be denoted by the fraction $\frac{1}{240}$; and all intermediate sums between a penny and a pound could only be expressed by fractions of proportionate value, the numerators and denominators of which would frequently be high numbers.

Such inconvenience would be unavoidable under any circumstances in which one unit, and one only, could be adopted. They are avoided by the use of several units of different magnitudes; so that when small sums are to be expressed, small units are used, the higher units being resorted to for higher sums. Also in the higher sums, where it is necessary to express fractions of the higher units, instead of such fractions, numbers of equivalent value composed of the inferior units are used. These observations, which are applicable generally to all complex numbers, will be more clearly apprehended when we proceed further with this subject.

The complex numbers which are used in the ordinary affairs of life are those which are necessary to express **TIME, MONEY, SPACE, and WEIGHT.**

Space is expressed by different kinds of complex numbers, according to the way in which it is considered. There are *measures of length*, *measures of surface*, and *measures of capacity*.

We shall now proceed to explain the different classes of units, and their mutual relation as to magnitude in these several species of complex numbers.

MEASURES OF TIME.

(300.) A **DAY** is that interval of time which elapses between two successive passages of the centre of the sun over the same point of the heavens. This interval being divided into 24 equal parts, each of these parts is called an **HOUR**.

An HOUR being divided in 60 equal parts, each of these parts is called a MINUTE.

A MINUTE being divided into 60 equal parts, each of these parts is called a SECOND.

A SECOND is the smallest subdivision of time used for the ordinary purposes of life, but for the more exact purposes of science, a second is again supposed to be subdivided into tenths.

Hours are expressed by placing the letter ^h above the number ; minutes, by placing ^m or ' above it ; and seconds, by ^s or '' : thus, 22 hours 35 minutes and 56 seconds would be expressed — $22^h\ 35^m\ 56^s$, or $22^h\ 35'\ 56''$.

(301.) Hence it appears, that when a number expresses days, it can be converted into an equivalent number of hours by multiplying it by 24 ; and a number expressing hours may be converted into days, or fractions of a day, by dividing it by 24.

(302.) When a number expresses hours, it may be converted into an equivalent number expressing minutes, by multiplying it by 60 ; and a number expressing minutes may be converted into an equivalent number expressing hours, or fractions of an hour, by dividing it by 60.

In the same manner, numbers expressing minutes may be converted into numbers expressing seconds, and *vice versâ*.

(303.) Since there are 24 hours in a day, and 60 minutes in an hour, we shall find the number of minutes in a day by multiplying 24 by 60 ; the product is 1440.

A number expressing days, therefore, may be converted into an equivalent one expressing minutes, by multiplying it by 1440 ; and a number expressing minutes may be converted into an equivalent one expressing days, or fractions of a day, by dividing it by 1440.

(304.) Since there are 60 minutes in an hour, and 60 seconds in a minute, we shall find the number of seconds in an hour by multiplying 60 by 60 ; the product is 3600.

A number expressing hours may therefore be converted into an equivalent one expressing seconds, by multiplying it by 3600; and a number expressing seconds may be converted into an equivalent one expressing hours, or fractions of an hour, by dividing it by 3600.

(305.) Since there are 1440 minutes in a day, and 60 seconds in a minute, we shall find the number of seconds in a day by multiplying 1440 by 60; the product is 86400.

A number expressing days may therefore be converted into an equivalent one expressing seconds, by multiplying it by 86400; and a number expressing seconds may be converted into an equivalent one expressing days, or fractions of a day, by dividing it by 86400.

(306.) A common year consists of 365 days, and a leap or bissextile year of 366 days. Every fourth year, commencing from the birth of Christ, is a leap or bissextile year, and every other a common year. Thus, if the number expressing any year from the birth of Christ be divided by 4, that year will be a leap year if there be no remainder, but otherwise it will be a common year. Thus 1832, being divided by 4, gives the quotient 458 without a remainder; the year 1832 was therefore a leap year. If 1834 be divided by 4, there will be a remainder, 2; 1834 is therefore a common year. There are, however, exceptions to this rule: the years which complete centuries from the birth of Christ are leap years only when their first two figures are divisible by 4 without a remainder. Thus, of the following, those only which are marked * are leap years: 1600*, 1700, 1800, 1900, 2000*, 2100, 2200, 2300, 2400*, &c. &c.

(307.) The results of the above calculations are expressed in the following

TABLE OF TIME.

1 minute = 60 seconds.

1 hour = 60 minutes = 3600 seconds.

1 day = 24 hours = 1440 minutes = 86400 seconds.

1 common year = 365 days.

1 leap year = 366 days.

Besides the above divisions of time, there are some others in ordinary use, such as *weeks*, *months*, *centuries*. A *week* is composed of 7 days, and a number expressing weeks is therefore converted into one expressing days by multiplying it by 7; a number expressing days is converted into one expressing weeks, and fractions of a week, by dividing it by 7.

The word *month* is used in different senses: it is sometimes used to express 4 weeks; in this sense it is distinguished from the *calendar month*, which is an interval of time varying in length. The year consists of 12 calendar months, some of 30, and others of 31 days; one month alone having 28 days, except in leap year, when it has 29. The months which have 30 days are April, June, September, and November. February has 28 days in common years, and 29 in leap years. The remaining months have 31 days.*

MONEY.

(308.) The classes of units by which sums of money are expressed are denominated *pounds*, *shillings*, and *pence*: the fractions of a penny in use are the half and the fourth, called the *halfpenny* and *farthing*. The relative value of these units is expressed in the following

TABLE OF MONEY.

1 penny = 2 halfpence = 4 farthings.

1 shilling = 12 pence = 24 halfpence = 48 farthings.

1 pound = 20 shillings = 240 pence = 480 halfpence = 960 farthings

The gold coin called a *sovereign* has the value of a pound, or 20 shillings; it contains 123.274 grains of standard gold: the purity of standard gold is in the proportion of 11 parts of pure gold to 1 of alloy: thus, a sovereign contains 113.001 grains of pure gold and 10.273 grains of alloy.

* The following well-known lines serve as a help to the memory to retain the length of the several months:—

Thirty days hath September,
April, June, and November;
February hath twenty-eight alone,
And all the rest have thirty-one.

The alloy contained in coin has no sensible value compared with the value of the coin of which it forms a part. It is used in the coin, or rather allowed to remain mixed with the precious metal, merely to save the expense which would be incurred in rendering the metal perfectly pure by refining. It is also of some use in rendering the metal harder, and more slow to wear.

A pound Troy weight of standard gold would, according to the weight just assigned to the sovereign, be coined into $46\frac{4}{9}$ sovereigns, which would be worth 46*l.* 14*s.* 6*d.* The value of an ounce Troy weight of standard gold will therefore be found by taking the twelfth part of this sum, which is 3*l.* 17*s.* 10½*d.*

THE SHILLING is a silver coin containing 80·727 grains of fine silver and 6·543 grains of alloy.

PENCE, HALFPENCE, and FARTHINGS are copper *tokens* not possessing the intrinsic value of the sums for which they are legally exchangeable. Thus, a silver shilling has a value 72 per cent. greater than 12 copper pennies. The evil effects which would result from this circumstance are counteracted by the manufacture of these tokens being confined to the government, and the restrictive condition that they are not a legal tender to an extent beyond one shilling in any single payment.

Formerly was current a gold coin denominated a guinea, and others of one half and one third of its value. The guinea was worth 21*s.* and the other two coins worth 10*s.* 6*d.* and 7*s.* These coins are now, however, out of use.

MEASURES OF SPACE.

(309.) To establish a uniform system of measures and weights, and to give such a system permanency, is an object of such general convenience and utility, that it could not fail to attract the attention of every nation at all advanced in civilisation; and we accordingly find that there is hardly a country in which some attempts to accomplish this have not been made. The attainment of such an end is, however, attended with many

practical difficulties, arising out of the very circumstances which render it desirable. The denominations of measure and weight have been necessarily in such constant and early use in domestic economy and in commerce, that each local system becomes deeply rooted from the effects of ancient custom, and is so intimately associated with the daily habits of life, that any attempt to change it is attended with almost as much difficulty as to change the general habits, manners, or language of a country. Until very recently, in different parts of Great Britain, the greatest confusion and inconvenience arose, from the total want of uniformity in the systems of weights and measures in common use. Different denominations were used in different parts of the kingdom; and still more frequently, and with greater inconvenience, the same denomination was used to express different quantities in different places; nay, even the same expressions were not unfrequently used, in the same place, to denote different quantities of different commodities. Thus, a stone of one commodity had a different weight from a stone of another, a gallon of one liquid had a different measure from a gallon of another, and so on.

It was attempted in various acts of parliament to remedy this inconvenience, but without effect, until the statute passed in 5 George IV., by which a uniform system of weights and measures was established, under the denomination of IMPERIAL WEIGHTS AND MEASURES, and their use enforced under severe penalties. This act has been generally enforced; and the system used throughout the British islands is now approaching to the desired uniformity.

To render this system permanent, standards of length and weight have been selected, which can at any future period be verified; so that if the weights or measures in use underwent in process of time any gradual alteration, however small, the amount of such variation might be ascertained, and a correction made with certainty and accuracy.

The standard or original unit of measure selected for

this purpose was the **YARD**; and its length was defined by expressing its numerical proportion to the length of a pendulum, which in the latitude of London and at the level of the sea vibrates seconds. Since the length of such a pendulum can always be ascertained, the length of the yard, which bears to it a fixed proportion, may always be verified.

Suppose the length of the seconds' pendulum to be divided into 391392 equal parts; 360000 of these parts are taken as the length of the yard. This length was selected, from the circumstance of its being as nearly as possible equivalent to that of the standard yard used previously to the statute now alluded to, by which means the original measure to which the public was accustomed underwent no apparent change; the effect being merely to provide the means of correcting its length at all future times, in the event of the original or standard yard having been destroyed, lost, or injured.

The *foot* is defined by the statute to be one third part of this standard yard; the *inch*, one twelfth of the foot. The *pole*, or *perch*, is defined to be $5\frac{1}{2}$ such yards; the *furlong*, 220 yards; and the *mile*, 1760 yards.

(310.) The following table exhibits the relative values of the different units in the system of Imperial measures of length: —

IMPERIAL MEASURES OF LENGTH.

1 foot = 12 inches.

1 yard = 3 feet = 36 inches.

1 perch = $5\frac{1}{2}$ yards = $16\frac{1}{2}$ feet = 198 inches

1 furlong = 40 perches = 220 yards = 660 feet = 7920 inches.

1 mile = 8 furlongs = 320 perches = 1760 yards = 5280 feet = 63360 inches.

To which we may add the following: —

1 league = 3 miles.

1 degree = $69\frac{1}{2}$ common miles.

1 geographical mile = $\frac{1}{80}$ th of a degree.

12 lines = 3 barleycorns = 1 inch.

1 palm = 3 inches.

1 hand = 4 inches.

1 span = 9 inches.

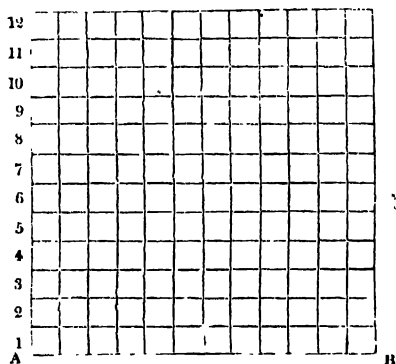
1 fathom = 6 feet.

The inch is sometimes divided into tenths, but more

commonly into twelfths, or lines. A further subdivision is sometimes made; the line being divided into 12 equal parts called seconds, and the seconds again into 12 equal parts called thirds. This duodecimal division is generally used in the measurement of artificers' work.

(311.) The extent of surfaces is measured by *square feet*, *square yards*, *square miles*, &c. A **SQUARE** is a figure formed by four straight lines equal in length, and placed so as to form right angles; the four straight lines forming such a figure are called the sides of the square, and the magnitude of the square is expressed by the length of its side. Thus, a square, whose side measures 1 inch, is called a *square inch*; one whose side measures 1 foot, is called a *square foot*, and so on.

As any length is expressed by the number of inches, feet, or miles it contains, so any surface is expressed by the number of square inches, square feet, or square miles it contains.



If 12 square inches be placed one beside the other in a horizontal row, A B, as here, twelve such rows, 1, 2, 3, 4, &c., placed one over the other, will form a square foot. This will be evident, since the height of the figure is equal to its breadth; each being 12 inches: the number of square inches in a square foot will therefore be found, by taking 12 square inches 12 times, or by

multiplying 12 by 12. One square foot, therefore, contains 12 times 12, or 144, square inches.

Since a yard consists of three feet, the same reasoning will show that there are 9 square feet in a square yard; and by pursuing a similar reasoning, we may obtain the results contained in the following table:—

SQUARE MEASURE.

Square. Square.

1 foot = 144 inches.

1 yard = 9 feet = 1296 inches.

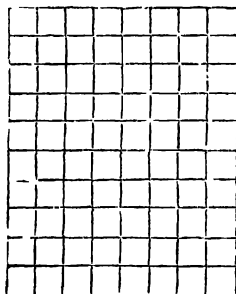
1 perch = 30½ yards = 272½ feet = 39204 inches.

1 rood = 40 perches = 1210 yards = 10890 feet = 1568160 inches.

1 acre = 4 roods = 160 perches = 4840 yards = 43560 feet = 6272640 inches.

1 mile = 640 acres.

(312.) If a space be bounded by four straight lines at right angles to each other, we shall find the number of square inches it contains, by multiplying the number of inches in its length by the number of inches in its breadth. This will be easily understood by reasoning similar to that already applied to a square foot. Suppose that the length is 10 inches, and the breadth 8; let a row of 8 square inches be formed, and let 10 such rows be placed one above the other, as in the figure:—



We shall thus have a space bounded by two straight lines at right angles to each other, one measuring 10 and the other 8 inches, and the whole consisting of 10 rows of 8 square inches. The total number of square inches is evidently 10 times 8, or 80.

The methods of finding the superficial magnitudes of

spaces bounded by straight lines not at right angles, or by curved lines, are explained in Geometry ; but are, in general, too complicated and difficult to be introduced here.

(313.) When we wish to express the magnitude of any solid body, if its sides be flat and at right angles to each other, we do so by declaring its length, breadth, and thickness ; but if it be of any other shape, its magnitude cannot be expressed in this way. It is not always sufficient to declare its length, breadth, and thickness, because by so doing, without some further calculation, the relative magnitudes of different bodies would not be known

The magnitudes of solids, in general, are expressed in a manner analogous to that already explained for expressing the magnitude of surfaces.

A *CUBE* is a figure of the shape of a common die with which we play in games of chance : it is bounded by 6 flat sides, forming 12 rectangular edges, all of the same length : its magnitude is denominated according to the length of its edge : if the length measure an inch, the solid is called a *cubic inch* ; if it measure a foot, it is called a *cubic foot*, &c. It is evident that each of the 6 sides of a cube is a square ; if it be a cubic inch, its sides are square inches ; if it be a cubic foot, its sides are square feet, &c.

From what has been already shown, it will follow that 12 rows of 12 cubic inches, placed on the same flat surface, will cover a square foot : now, if 12 layers of these be piled one upon the other, we shall have a solid 12 inches in height, 12 inches in breadth, and 12 inches in thickness ; we shall, in fact, have a *cube* whose edge measures a foot. A cubic foot, therefore, may be considered as consisting of 12 layers of cubic inches, each layer consisting of 12 rows, having 12 cubic inches in each row.

It is evident, therefore, that to find the number of cubic inches which form a cubic foot, we must first multiply 12 by 12, and then multiply the product by 12 ; the result will be the cube of 12, which is 1728. In the same manner, if the edge of a cube consist of 3

feet, it will contain a number of cubic feet expressed by the cube of 3 : a cubic yard, therefore, consists of 27 cubic feet.

If a solid body be bounded by flat surfaces at right angles to each other, we shall always find the number of cubic inches it contains, by ascertaining the number of inches in its length, breadth, and height : multiply the number of inches in its length by the number of inches in its breadth, and multiply the product by the number of inches in its height ; the result will be the number of cubic inches in the solid. This may be easily understood, by considering how the solid may be built with cubic inches. Let us suppose that its length is 10 inches, its breadth 8, and its height 5 ; place 8 cubic inches in a horizontal row, and then place 10 of these rows one beside the other : we shall thus form a solid, whose length is 10 inches, breadth 8 inches, and height 1 inch ; let 5 such layers be piled one upon the other, and we shall form a solid of the same length and breadth, but with the height of 5 inches. Now, since each layer consists of 10 rows of 8 cubes, the number of cubes in each layer will be found by multiplying 10 by 8 ; and since there are 5 such layers in the solid so formed, we shall get the total number of cubes by multiplying the number in each layer by 5. We thus multiply the length by the breadth, and then multiply the product by the height.

(314.) The following table exhibits the relative values of the different units of cubic measure.

CUBIC MEASURE.

1728 cubic inches	-	-	= 1 cubic foot.
27 cubic feet	-	-	= 1 cubic yard.
40 feet of rough timber, or	}	= 1 load, or ton.	
50 feet hewn ditto			
42 cubic feet	-	-	= 1 ton of shipping.

By the denominations in this table, all artificers' solid work is measured, such as carpentry, masonry, and, in general, all works estimated by length, breadth, and thickness.

(315.) Liquids in general, and all substances which are in a state like that of powder, so as to be capable of filling a hollow vessel, are measured by a system of measures denominated "the Imperial liquid and dry measure." This system has for its basis, or primary unit, the Imperial gallon, which is a vessel, the capacity of which is $277\frac{1}{4}$, or $277\frac{1}{2}$, cubic inches very nearly. The manner in which the magnitude of this vessel is defined by the act, is that it shall be such as to contain 10 pounds avoirdupois weight of distilled water, weighed in air at the temperature of 62° of Fahrenheit, the barometer being at 30 inches. It is declared that such Imperial standard gallon shall be the only standard measure of capacity to be used for wine, beer, and all sorts of liquids, and for such dry goods as are measured like liquids and not heaped; that from this all other measures shall be derived; the quart to be a fourth part of the gallon, the pint an eighth part; and 2 such gallons to be a peck, 8 to be a bushel, and 8 bushels to form a quarter of corn, or other dry goods not sold by heaped measure. The following table exhibits the relative and absolute values of the different classes of units forming the Imperial liquid and dry measure not heaped:—

IMPERIAL LIQUID AND DRY MEASURE.

Gals.	Pint.	Quart.	Pottle.	Gallon.	Peck.	Bushel.	Coom.
4 =	1						
8 =	2 =	1					
16 =	4 =	2 =	1				
32 =	8 =	4 =	2 =	1			
64 =	16 =	8 =	4 =	2 =	1		
256 =	64 =	32 =	16 =	8 =	4 =	1	
1024 =	256 =	128 =	64 =	32 =	16 =	4 =	1
2048 =	512 =	256 =	128 =	64 =	32 =	8 =	2 = 1

When heaped measure is used, the bushel is directed by the act to be constructed in the following manner:—It shall contain 80 pounds avoirdupois of distilled water, being made cylindrical, and having a diameter not less than double its depth; that the goods measured shall be heaped in the form of a cone, the height of which shall be at least 3 fourths of the depth of the measure, and the outside of the bushel to be the ex-

tremitry of the base of the cone : 3 such bushels shall be a sack ; and 12 sacks, a chaldron. The contents of the Imperial heaped bushel amount to 2815·4887 cubic inches.

(316.) As there is frequently occasion to refer to the old systems of measures, which have been superseded by the Imperial system, the following tables exhibiting the relative and absolute values of those measures may be useful.

OLD WINE MEASURE.

4 gills	=	1 pint.
2 pints	=	1 quart.
4 quarts	=	1 gallon.
42 gallons	=	1 tierce.
2 tierces	=	1 puncheon.
63 gallons	=	1 hogshead.
2 hogsheads	=	1 pipe or butt.
2 pipes	=	1 tun.

From this, and the table of Imperial measure, we derive the following results : —

(317.) WINE MEASURE.

	Imp. galls.	Imp. galls.
1 tierce	= 34·99066	= 35 nearly.
1 puncheon	= 69·98132	= 70 nearly.
1 hogshead	= 52·48599	= 52½ nearly.
1 pipe	= 104·97198	= 105 nearly.
1 tun	= 209·94396	= 210 nearly.

(318.) OLD ALE AND BEER MEASURE.

4 gills	=	1 pint.
2 pints	=	1 quart.
4 quarts	=	1 gallon.
8 gallons	=	1 firkin of ale.
9 gallons	=	1 ditto of beer.
2 firkins	=	1 kilderkin.
2 kilderkins	=	1 barrel.
1½ barrel	=	1 hogshead.
2 barrels	=	1 puncheon.
2 hogsheads	=	1 butt.
2 butts	=	1 tun.

(319.) OLD DRY OR WINCHESTER MEASURE.

4 gills	= 1 pint.
2 pints	= 1 quart.
2 quarts	= 1 pottle.
2 pottles	= 1 gallon.
2 gallons	= 1 peck.
2 pecks	= 1 bushel.
4 bushels	= 1 coom.
2 cooms	= 1 quarter.
5 quarters	= 1 wey, or load.
2 weys	= 1 last.

(320.) Since the old wine gallon contains 231 cubic inches, and the Imperial gallon 277·274, any number expressing wine gallons will be converted into an equivalent number of Imperial gallons, by diminishing it in the proportion of 277·274 to 231, or, what is the same, in the proportion of 277274 to 231000: this reduction may be made by multiplying the number of wine gallons by the fraction $\frac{231000}{277274}$, or by its equivalent decimal, 0·83311.

To convert a number expressing Imperial gallons into an equivalent one expressing wine gallons, we should, on the contrary, increase it in the proportion of 231000 to 277274, which may be done by multiplying it by the fraction $\frac{277274}{231000}$, or by its equivalent decimal, 1·20032.

When extreme accuracy is not required, a number expressing wine gallons may be reduced to an equivalent number of Imperial gallons, by multiplying it by 5, and dividing the product by 6; and a number expressing Imperial gallons may be reduced to an equivalent number of wine gallons by the contrary process, viz. multiplying by 6, and dividing by 5.

Since the old ale gallon contains 282 cubic inches, and the Imperial standard gallon 277·274 cubic inches, any number expressing ale gallons will be converted into an equivalent number expressing Imperial gallons, by diminishing it in the proportion of 282 to 277·277, or, what is the same, in the proportion of

282000 to 277274: this would be done by multiplying it by the fraction $\frac{282000}{277274}$, or by the equivalent decimal, 1·0170445. To convert a number expressing Imperial gallons into an equivalent number expressing ale gallons, it would be necessary to diminish it in the proportion of 282000 to 277274, which would be done by multiplying it by the fraction $\frac{277274}{282000}$, or by the equivalent decimal, 0·9882411.

When extreme accuracy is not necessary, the number expressing ale gallons may be reduced to an equivalent number expressing Imperial gallons, by multiplying it by 60, and dividing the product by 59; and a number expressing Imperial gallons may be converted into an equivalent one expressing ale gallons, by multiplying it by 59 and dividing the product by 60.

Since the Winchester bushel contains 2150·420 cubic inches, and the Imperial bushel 2218·198 cubic inches, any number expressing Winchester bushels may be converted into an equivalent one expressing Imperial bushels, by diminishing it in the proportion of 2218192 to 2150420, which will be done by multiplying it by the fraction $\frac{2150420}{2218192}$, or by its equivalent decimal, 0·969447. A number expressing Imperial bushels may be converted into an equivalent one expressing Winchester bushels, by multiplying it by the fraction $\frac{2218192}{2150420}$, or by its equivalent decimal, 1·0315157.

When great accuracy is not required, a number expressing Winchester bushels may be reduced to one expressing Imperial bushels, by multiplying it by 31, and dividing the product by 32; and a number expressing Imperial bushels may be converted into one expressing Winchester bushels, by multiplying it by 32, and dividing the product by 31.

It would be much more convenient and equitable, if grain, seeds, and all substances which are usually sold by heaped measure, were sold by weight: the weight of a substance is always proportional to its quantity; but the same measure of different specimens of the same commodity will differ in quantity. The average bushel

of wheat is generally considered as equivalent to 60 pounds weight; but wheat of different qualities will vary from this medium, a bushel being sometimes less and sometimes greater than 60 pounds: the average weight of a bushel of barley is reckoned at 47 pounds; of oats, 38 pounds; of peas, 64 pounds; of beans, 63 pounds; of clover, 68 pounds; of rye and canary, 53; and of rape, 48.

Coals, which were formerly sold by measure, are now sold by weight.

MEASURES OF WEIGHT.

(321.) We have seen that the system of measures of length, surface, and capacity are all derived, by definite numerical proportions, from the length of a pendulum which vibrates seconds, by reference to which the system of measures may be verified and corrected at all future times. In order to obtain a similar test for the verification of weights, the imperial standard pound troy weight was compared with the weight of a cubic inch of distilled water, at a fixed temperature and under a fixed atmospheric pressure. The weight of such quantity of water being supposed to be divided into 505 equal parts, each of these parts would be half a grain troy weight. The pound troy* was declared to be composed of 5760 grains thus determined, and the pound avoirdupois† to be composed of 7000 such grains.

* "There are reasons to believe that the word *troy* has been derived from the monkish name given to London of *Troy novant*, founded on the legend of Brute. Troy weight, therefore, according to this etymology, is in fact London weight. We were induced, moreover, to preserve the troy weight, because all the coinage has been uniformly regulated by it, and all medical prescriptions of formulæ now are, and always have been, estimated by troy weight under a peculiar subdivision, which the College of Physicians have expressed themselves most anxious to preserve."—*Report of Commissioners of Weights and Measures.*

† "We find the avoirdupois weight by which all heavy goods have been for a long time weighed (probably derived from *avoirs* (*averia*) the ancient name for GOODS or CHATTELS, and *poids*, WEIGHT,) to be universally used throughout the kingdom. This weight, however, seems not to have been preserved, with such scrupulous accuracy as troy weight, by which the most precious articles have been weighed; but we have reason to believe that the pound cannot differ by more than 1, 2, or 3 grains from 7000 grains troy. It therefore occurred to us that we should be offering no violence to

The relative values of the units of imperial troy weight are expressed in the following table: —

(322.) — TABLE OF TROY WEIGHT.

1 pennyweight (dwt.)	= 24 grains.
1 ounce (oz.)	= 20 pennyweights.
1 pound (lb.)	= 12 ounces.

This denomination of weight is used in estimating the quantity of the precious metals, jewels, &c. It is also used in determining specific gravities, and in general in philosophical investigations. For the purposes of excise it is used in determining the strength of spirituous liquors.

The troy weight, differently divided, is used by apothecaries for compounding prescriptions.

(323.) — APOTHECARIES' WEIGHT.

1 scruple \mathfrak{S}	= 20 grains.
1 dram \mathfrak{z}	= 3 scruples.
1 ounce \mathfrak{z}	= 8 drams.
1 pound \mathfrak{lb}	= 12 ounces.

(324.) A peculiar division of weight is used in estimating the value of diamonds: $3\frac{1}{5}$ grains troy weight make 1 carat; 1 carat is divided into 4 equal parts called *grains*, and the grain is resolved into 16 equal divisions called *parts*: thus one part diamond weight is the twentieth part of a grain troy weight. The relative value of the units avoirdupois weight are expressed in the following table: —

(325.) — AVOIRDUPOIS WEIGHT.

1 scruple	= 10 grains.
1 dram	= 3 scruples.
1 ounce	= 16 drams.
1 pound	= 16 ounces.
1 quarter	= 28 pounds.
1 cwt.	= 4 quarters.
1 ton	= 20 cwt.

this system of weights, if we declared that 7000 grains troy should hereafter be considered as the pound avoirdupois." — *Same Report.*

From a comparison of this table with the table of troy weight, it will appear that 7680 grains avoirdupois = 7000 grains troy, each of these being equal to a pound. One grain troy weight is, therefore, equal to 1.097 grains avoirdupois.

It follows, also, that while the ounce avoirdupois is less than the ounce troy, in the proportion of 175 to 192, the pound avoirdupois is greater than the pound troy, in the proportion of 175 to 144. Thus

$$\begin{array}{rcl} 144 \text{ lbs. avoirdupois} & = & 175 \text{ lbs. troy.} \\ 192 \text{ oz.} & \text{————} & = 175 \text{ oz.} \end{array}$$

In general a stone weight is 14 pounds avoirdupois, but for butcher's meat or fish it is 8 pounds: the hundred weight, therefore, is 8 stone of 14 pounds, or 14 stone of 8 pounds. In weighing glass the stone is 5 pounds; and a denomination of weight is used called a seam, which is 24 stone, or 120 pounds. In weighing hay, the truss weighs half a hundred weight, or 56 pounds. If the hay be new, which it is considered to be until the 1st of September in the year in which it is grown, the truss is taken to be 60 pounds. The truss of straw is 36 pounds.

(326.) The divisions of avoirdupois weight by which wool is estimated are expressed in the following table:—

WOOL WEIGHT.

7 pounds	=	1 clove.
2 cloves	=	1 stone.
2 stone	=	1 tod.
$6\frac{1}{2}$ tods	=	1 wey.
2 weys	=	1 sack.
12 sacks	=	1 last.
1 pack	=	240 pounds.

(327.) In weighing cheese and butter, the following denominations are used:—

CHEESE AND BUTTER WEIGHT.

8 pounds	=	1 clove.
32 cloves	=	1 wey (Essex).
42 cloves	=	1 wey (Suffolk).
56 pounds	=	1 firkin.
q 2		

FRENCH SYSTEM.

(328.) No nation has succeeded in establishing a system of weights, measures, and money, at once so simple and uniform as that which has been established since the revolution in France. The basis on which this system is founded is the magnitude of a quadrant, or fourth part of the meridian; that is, the distance from the equator to the pole, as it would be measured upon the surface of the sea uninterrupted by the irregularities of land. This measurement was made with the utmost accuracy by observations on arches of the meridian in different latitudes, by which the exact figure of the meridian was ascertained, and the actual length of an arch of considerable magnitude, extending from north to south between Dunkirk and Barcelona. The whole length of the meridional quadrant being divided into 10000000 equal parts, one of these parts was taken as the primary and original unit from which all weights and measures should be derived: it was called the *METRE*, and its length corresponded very nearly with the ancient French yard, being equal to 3·07844 French feet, or 3·281 English feet, or 39·3708 English inches.*

The object kept in view in the formation of the French weights and measures was to enable all quantities to be expressed by whole numbers and decimals, without the separation into classes of units distinct from the local values which the digits have in the ordinary arithmetical notation: this was accomplished by forming the classes of units of weight, measure, and money according to the decimal scale.

The metre being the original unit, the next superior unit was the *decametre*, which was equivalent to 10 metres; next above this was the *hectometre*, or 100 metres, and then followed the *kilometre*, or 1000 metres, and the *myriametre*, or 10000 metres. In general,

* By an accidental correspondence, the metre is very nearly equal to the length of the seconds' pendulum.

the successive units rising in a decuple progression above the metre were expressed by Greek prefixes ; Latin prefixes were adopted to express the classes of units inferior to the metre : thus the tenth part of the metre was called a *décimetre*, the hundredth part the *centimetre*, the thousandth the *millimetre*.

The convenience of this system will be evident if we attempt to express by it any length in which units of several classes occur : thus, suppose we have to express 6 myriametres, 5 kilometres, 3 decametres, 4 metres, 6 decimetres, 8 centimetres, and 9 millimetres. Had the relative values of these units not been formed on the decimal scale, we could only express them as so many distinct numbers, writing the name of each unit above the numbers respectively, as we do in English measures, with miles, furlongs, perches, &c. But the relation being formed on the decimal system, the above length may be expressed thus : — 65034·689 metres.

It is evident, that all lengths, however numerous the classes of units they contain, can be expressed by simple numbers, the units inferior to the metre occupying the decimal places, and the superior units those of tens, hundreds, &c. All arithmetical operations may, therefore, be performed on such numbers according to the rules already established for whole numbers and decimals.

(329.) The lengths of the several denominations of French measures of length are expressed in English measures in the following table : —

Millimetre	=	0·03937	inches.
Centimetre	=	0·393708	—
Decimetre	=	3·937079	—
Metre	=	39·37079	—
		3·2808992	feet.
		1·093633	yards.
Myriametre	=	6·2138	miles.

In the following table, the English measures are expressed in terms of the French measure : —

1 inch	=	{ 2·539954 centimetres.
1 foot	=	3·0479449 decimetres.
1 imperial yard	=	0·91438348 metres.
1 fathom (2 yards)	=	1·82876696 ———
1 perch	=	5·02911 ———
1 furlong	=	201·16437 ———
1 mile	=	1609·3149 ———

(330.) The French system of superficial measures is derived from that of linear measure; the *are* is the unit of superficial measure, and is equal to 100 square metres; the *centiare* is 1 square metre, and the *hectare* is 100 ares, or 10000 square metres. The following tables will serve for the reduction of French to English measures, and vice versâ:—

Square metre	=	1·196033 square yards.
Are	=	0·098845 roods.
Hectare	=	2·473614 acres.
Square yard	=	0·836097 square metres.
perch	=	25·291939 ———
Rood	=	10·116775 ares.
Acre	=	0·404671 hectares.

(331.) The units of solid measure are the *stere*, or cubic metre, and the *decistere*, which is a tenth of the former. There are three denominations of liquid measure, the *litre*, which is a cubic decimetre; the *decalitre*, which is 10 decimetre cubes, and the *decilitre*, which is the tenth part of a decimetre cube.

The measures for dry goods are the *litre*, or decimetre cube, the *decalitre*, *hectolitre*, and *kilolitre*, which are respectively for 10, 100, and 1000 decimetre cubes. The relation between these and English measures is exhibited in the following tables:—

Litre	=	{ 1·760773 pints.
		0·2200967 gallons.
Décalitre	=	2·2009667 gallons.
Hectolitre	=	22·009667 gallons.

Pint	-	=	0.567932	litres.
Quart	-	=	1.135864	—
Imperial gallon	=		4.54345794	—
Peck	-	=	9.0869159	—
Bushel	-	=	36.347664	—

(332.) The basis of the French system of weights is the *kilogramme*, which is the weight of a decimetre cube of distilled water at the temperature of 40° of Fahrenheit's thermometer: the thousandth part of this, or the *gramme*, is the unit of weight: the *decagramme*, *hectogramme*, and *kilogramme* are respectively 10, 100, and 1000 grammes. The *quintal* is 100 kilogrammes, and the *milier* 1000 kilogrammes; the *decigramme* is the tenth part of the gramme; the *centigramme* the hundredth part of the gramme, and so on. The following tables express, very nearly, the relation between the French and English weights: —

Gramme	-	=	{ 15.438 grains troy.
			{ 0.643 pennyweights.
			{ 0.03216 ounces troy.
Kilogramme	-	=	{ 2.68027 pounds troy.
			{ 2.20548 — avoird.

Troy wt.				
1 grain	-	=	0.06477	grammes.
1 pennyweight	=		1.55456	—
1 ounce	-	=	31.0913	—
1 imperial pound	=		0.3730956	kilogrammes.

Avoirdupois.				
1 dram	-	=	1.7712	grammes.
1 ounce	-	=	28.3384	—
1 imperial pound	=		0.4534148	kilogrammes.
1 hd. weight	-	=	50.78246	—
1 ton	-	=	1015.649	—

(333.) The unit of French money is the silver coin called a **FRANC**: it consists of 9 parts of pure silver and 1 of alloy. The weight of a 5 franc piece is 25 grammes; so that 5 grammes of standard silver represent the value of 1 franc: the franc is supposed to be divided into 10 parts called *decimes*; the decime, again,

is divided into 10 parts, called *centimes*. Any sum of money, however great, is expressed in francs and decimal parts of a franc, so that all calculations of money are made by the rules established for whole numbers and decimals.

An English pound sterling, when the exchange is at par, is equivalent to 25·2 francs; thus the value of a franc is 9·523 pence, or very nearly $9\frac{1}{2}d$.

(334.) The operation which, in treatises on arithmetic, is generally distinguished by the name **REDUCTION** is that by which complex numbers are converted into simple numbers, or vice versa; or more generally, by which the numbers expressing one class of units are converted into equivalent numbers expressing other classes. The relative values of the different classes of the units of complex numbers exhibited in the preceding part of this chapter furnish all the data necessary for such reductions; but as these operations afford a useful exercise in the arithmetical principles already developed, and in many cases give rise to rules and methods of calculation which are useful in the ordinary affairs of life, we shall here enter into the details of some of the most necessary and useful of such calculations. The spirit of the method by which they are conducted will be readily seized by the student, and applied to other cases of complex numbers, for which we cannot conveniently afford space.

Let it be required to convert the sum of 17*l.* 16*s.* $9\frac{3}{4}d$. into an equivalent number of farthings. We may proceed to accomplish this by multiplying the pounds by 960, the shillings by 48, and the pence by 4; these being respectively the number of farthings contained in a pound, a shilling, and a penny: but it is generally more convenient to convert the pounds into shillings in the first instance, then add the shillings to the result. and convert the total number of shillings into pence: to the pence thus obtained, add the pence in the given sum, and convert the whole into farthings, adding the farthings in the given sum. The process would be as follows:—

	\pounds	<i>s.</i>	<i>d.</i>
Multiply	17	16	$9\frac{1}{4}$
by	20		
	<hr/>		
	340	shillings.	
Add	\times 16	—	
	<hr/>		
Multiply	356	—	
by	12		
	<hr/>		
	4272	pence.	
Add	9	—	
	<hr/>		
Multiply	4281	—	
by	4		
	<hr/>		
	17124	farthings.	
Add	3	—	
	<hr/>		
	17127	farthings =	$\pounds 17$ 16 <i>s.</i> $9\frac{1}{4}$ <i>d.</i>

By multiplying 17 pounds by 20 we reduce them to shillings, to which we add the 16 shillings of the given sum: we thus find that the given sum consists of 356 shillings and 9 pence 3 farthings; we then convert the shillings into pence by multiplying by 12; and adding the 9 pence of the given sum, we find that the given sum consists of 4281 pence 3 farthings: converting this number of pence into farthings by multiplying it by 4, and adding the 3 farthings of the given sum, we find that the total number of farthings is 17127.

The practical process may be somewhat abridged by adding at once the shillings in the process of multiplying by 20, and in the same manner adding the pence and farthings in the process of multiplying by 12 and 4. It is also unnecessary to write the multipliers 20, 12, and 4, since they are always well known: the written process would then be as follows:—

\pounds	<i>s.</i>	<i>d.</i>
17	16	$9\frac{1}{4}$
<hr/>		
356		
<hr/>		
4281		
<hr/>		
17127		
<hr/>		

To convert a simple number expressing farthings into a complex one expressing pounds, shillings, &c. we divide successively by 4, 12, and 20. If the given number be 17127, the process would be as follows : —

$$\begin{array}{r}
 4 \overline{) 17127} \\
 \underline{12 \frac{3}{4}} \\
 20 \overline{) 356} \quad 9\frac{1}{2} \\
 \underline{\pounds 17 \quad 16 \quad 9\frac{1}{2}}
 \end{array}$$

When the number is divided by 4, the quotient will be pence, and the remainder farthings. This number of pence, divided by 12, will give a quotient expressing shillings, and a remainder expressing pence. This number of shillings, divided by 20, will give a quotient expressing pounds, and a remainder expressing shillings. We thus obtain the pounds, shillings, pence, and farthings in the proposed sum.

(335.) It is sometimes required to express a sum of money in pounds and decimals of a pound.

Since a shilling is the twentieth part of a pound, it will be expressed by 5 in the second decimal place, since $\frac{1}{20}$ is $\frac{5}{100}$: thus, 1 shilling = $\pounds 0.05$. We shall, therefore, find the decimal of a pound which is equivalent to any number of shillings, by multiplying 0.05 by the number of shillings ; or, what will produce the same result, let the number of shillings be multiplied by 5, and take 2 decimal places in the product. Thus, if the number of shillings be 12, the equivalent decimal of a pound will be 0.60 : if the number of shillings be 9, the equivalent decimal of a pound will be 0.45. The following is an easy practical rule for making this calculation : —

If the number of shillings be even, place half that number in the first decimal place ; and if it be odd, place in the first decimal place half the number which is one less than the shillings, and 5 in the second. Thus, if the number of shillings be 6, the equivalent decimal is 0.3 ; if the number of shillings be 16, the equivalent decimal is 0.8 ; if the number of shillings

be 17, the equivalent decimal is 0·85 ; and so on. If the number of shillings proposed exceed 20, it should first be reduced to pounds and shillings, and then the shillings reduced to decimals.

Let it be required to express 375 shillings in decimals of a pound. To reduce shillings to pounds, it is necessary to divide by 20. An abridged method of performing this operation may be obtained in the following manner : — We may divide by 20 by dividing by 10 and 2 successively. We shall divide by 10 by cutting off the units' figure of the shillings, and taking it as remainder, and the other figures as the quotient (185.). In the present case this quotient would be 37, and the remainder 5 ; but since we have divided the shillings by 10, each unit of this quotient, 37, signifies 10 shillings. If we divide it by 2, we shall obtain the quotient 18 and the remainder 1 : the quotient 18 will signify pounds, and the remainder 1 will express 10 shillings. Thus, 375 shillings will be equivalent to 18 pounds, 10 shillings and 5 shillings, or 18*l.* 15*s.* 0*d.*

In general, then, to reduce shillings to pounds, cut off the last figure, and divide the remaining figures by 2. The quotient will be the number of pounds, and if there be no remainder the figure cut off will be the number of shillings ; but if there be a remainder of 1, that 1 must be prefixed to the figure cut off to express the shillings. In the present example, when 37 was divided by 2, there was a remainder 1, which, prefixed to the 5 cut off, gave 15 shillings. Having obtained the number of pounds and shillings, we may now convert the shillings into a decimal of a pound by the rule already given, and we find that 15*s.* is equal to £ 0·75 ; and, therefore, 375*s.* = £18·75.

(336.) Since there are 960 farthings in a pound, we shall find the decimal of a pound, which is equivalent to a farthing, by converting the fraction $\frac{1}{960}$ into an equivalent decimal. This decimal is 0·0010416. In the same manner, we shall find the decimal of a pound equivalent to 2, 3, 4, &c. farthings, by converting the fractions $\frac{2}{960}$, $\frac{3}{960}$, $\frac{4}{960}$, &c. into equivalent decimals.

It will be found that in all such decimals the figures which fill the first three decimal places are those which express the number of farthings, provided that number be less than 24. For 24 farthings the equivalent fraction of a pound is $\frac{3}{80}$ or $\frac{1}{30}$; and the equivalent decimal is, therefore, 0.025. Whenever the number of farthings exceeds 23, then, it will be found that the figures which fill the first three places express a number which exceeds the proposed number of farthings by 1. This observation extends to every number of farthings less than 48; and, since 48 farthings are equivalent to a shilling, it is never necessary to seek the equivalent decimal for a greater number of farthings than 47. It appears, therefore, that we can always find the first 3 places of the decimal of a pound, which is equivalent to any number of farthings, by the following rule:—

If the number of farthings be less than 24, place the figure or figures which express them in the third, or in the second and third, decimal places. If the number be not less than 24, then add 1 to it, and proceed in the same manner. Since the decimal places beyond the third express fractions less than the thousandth part of a pound, the quantity they express is less than a farthing. Such places may, therefore, be omitted in all calculations where quantities less than a farthing are disregarded.

Example.—Let it be required to reduce $10\frac{3}{4}d.$ to an equivalent decimal of a pound. The number of farthings will be 43, and the equivalent decimal will be 0.044. If the sum be $8\frac{1}{2}d.$, the number of farthings will be 34, and the equivalent decimal will be 0.035, &c.

By combining this rule with that established for shillings, we shall be able, without difficulty, to find the decimal of a pound equivalent to any number of shillings and pence. Let the sum be $18s. 7\frac{3}{4}d.$, we find the decimal as follows:—

$$\begin{array}{rcl}
 & & \text{£} \\
 18s. \ 0d. & = & 0.90 \\
 7\frac{3}{4}d. & = & 0.082 \\
 & & \hline
 & & 0.982
 \end{array}$$

To convert 25*l.* 14*s.* 3½*d.* into an equivalent decimal of pounds: —

$$\begin{array}{rclcl}
 \text{£} & \text{s.} & \text{d.} & & \text{£} \\
 25 & 14 & 0 & = & 25\cdot70 \\
 & & 3\frac{1}{2} & = & 0\cdot014 \\
 & & & & \hline
 & & & & 25\cdot714
 \end{array}$$

(337.) By retracing this process in a reverse order, we shall obtain a rule for converting any decimal of a pound into shillings and pence.

To find the number of shillings, double the number which occupies the first decimal place, and if the number occupying the second place of decimals be not less than 5, add 1 to the number so found. This will be the number of shillings. If the number occupying the second decimal place be greater than 4, subtract 5 from it. If the number which remains prefixed to the figure in the third decimal place be less than 24, that number will be the number of farthings in the required number. If it be not less than 24, then it will be 1 more than the number of farthings. The number of farthings being thus found and divided by 4 will give the number of pence.

Let it be required to convert the decimal £75·876 into pounds, shillings, &c. We double the first decimal figure, 8, and obtain 16. The second place being greater than 5, we add 1 to 16, which gives 17 for the number of shillings. Subtracting 5 from the second place, we prefix the remainder to the third place, which gives 26. This being greater than 24, we subtract 1 from it, and find 25 for the number of farthings. This, divided by 4, gives 6¼*d.* The number sought is, therefore, 75*l.* 17*s.* 6¼*d.*

To convert a complex number, expressing a distance in miles, furlongs, perches, yards, feet, and inches, into a simple number expressing inches, we might multiply the units of each denominator by the number of inches

which they respectively contain (310.), and then add together the several products ; but the process generally used is, to reduce the miles to furlongs by multiplying them by 8, and add to the product the furlongs in the proposed length. The furlongs thus found are reduced to perches by multiplying them by 40, to which the number of perches are added ; and the reduction is continued in the same way, upon the principle already applied to sums of money.

Let it be required to convert into inches the following distance : —

miles.	furl.	perch.	yds.	ft.	inch.
17	6	22	4	2	7

the process will be as follows : —

m.	f.	p.	yds.	ft.	in.
17	6	22	4	2	7
8					
<hr/>					
142	furlongs.				
40					
<hr/>					
5702	perches.				
5.5					
<hr/>					
28510					
28510					
4					
<hr/>					
31365.0					
<hr/>					
31365	yards.				
3					
<hr/>					
94097	feet.				
12					
<hr/>					
1129171	inches.				

By multiplying by 8, and adding 6 to the product, we obtain the number of furlongs. Multiplying this by 40, and adding 22, we obtain the number of perches. To reduce this to yards, we should multiply by $5\frac{1}{2}$, or

by the equivalent decimal 5.5. Performing this multiplication, and adding 4 in the units' column for the yards in the given sum, we obtain the number of yards, omitting the 0 which fills the decimal place. We obtain the feet by multiplying the yards by 3, and adding the 2 feet in the given sum; and we obtain the inches by multiplying the feet by 12, and adding the 7 inches in the given sum.

By reversing this process, we may convert a simple number, expressing inches, into an equivalent one, expressing miles, furlongs, &c.

Let the proposed number be 1129171 inches. To reduce this to feet divide it by 12: we obtain the quotient 94097, with the remainder 7: the quotient is here feet, and the remainder inches. To reduce this number of feet to yards, we divide by 3, and obtain the quotient 31365, with the remainder 2, which remainder is yards. In the same manner we proceed dividing successively by 5.5, 40, and 8, and we obtain quotients and remainders corresponding to the numbers in the above process, until, finally, we reproduce the complex number with which we commenced.

To convert a number of tons, hundred weights, quarters, &c. into an equivalent number of ounces, we proceed in a manner altogether analogous to the methods already applied to other complex numbers; the relative values of the different classes of units being derived from the table (325.).

Let the proposed complex number be 15 tons, 7 cwt. 3 qrs. 23 lbs. 15 oz. We reduce the tons to hundred weights by multiplying by 20, adding 7 to the product. We reduce the hundred weights to quarters by multiplying by 4. In the same manner the quarters are reduced to pounds by multiplying by 28, and the pounds to ounces by multiplying by 16, adding to the several products the sums taken from the given complex number. The process at length will be as follows: —

	tons.	cwt.	qrs.	lbs.	oz.
	15	7	3	23	15
Multiply	20				
	300	cwt.			
Add	7	—			
	307				
Multiply	4				
	1228	qrs.			
Add	3	—			
	1231				
Multiply	28				
	9848				
	2462				
	34468	lbs.			
Add	23	—			
	34491				
Multiply	16				
	206946				
	34491				
	551856	oz.			
Add	15	—			
	551871				

The reverse process of converting a number expressing ounces into an equivalent number expressing tons hundred weights, &c. would be as follows:—

$$\begin{array}{r}
 16 \) \ 551871 \text{ oz.} \\
 \hline
 28 \) \ 34491 \text{ lb. } 15 \text{ oz.} \\
 \hline
 4 \) \ 1231 \text{ qrs. } 23 \text{ lb. } 15 \text{ oz.} \\
 \hline
 20 \) \ 307 \text{ cwt. } 3 \text{ qrs. } 23 \text{ lb. } 15 \text{ oz.} \\
 \hline
 15 \text{ tons, } 7 \text{ cwt. } 3 \text{ qrs. } 23 \text{ lb. } 15 \text{ oz.}
 \end{array}$$

In dividing by 16, there is a remainder 15, which signifies 15 ounces, the quotient giving the number of pounds. Dividing this by 28, we get the quarters, with a remainder 23, which signifies 23 pounds remain-

ing over and above quarters. The first remainder, 15 ounces, must still be brought down. Dividing again by 4, we get the number of hundred-weights, with 3 quarters remaining over ; and dividing the hundred-weights by 20, we get the tons, with a remainder over of 7 hundred-weights. This process is exactly the reverse of the first one.

It is unnecessary to pursue such reductions farther, since the methods applied to different classes of complex numbers have no other difference than that which arises from the difference between the relative values of their several units.

CHAP. II.

OF THE ADDITION AND SUBTRACTION OF COMPLEX NUMBERS.

(338.) THE method by which the several arithmetical operations are performed on complex numbers rests upon principles essentially identical with those which govern the same operations performed on simple numbers. In these numbers, as expressed by the common arithmetical notation, there is, in fact, a regular succession of distinct classes of units. The difference between them and complex numbers only consists in this, that in simple numbers each superior class of units has the same numerical relation to the class below it, the proportional values being always in a decuple progression; whereas, in complex numbers, that regular relation is not found to exist between the successive classes of units.

The operations of addition and subtraction, in simple numbers, are effected by performing them successively on each order of units, commencing from the units' column, and proceeding from right to left. The same method precisely is observed in complex numbers; and the difference between the two operations only arises from the way in which numbers must be *carried* from one order of units to another. These general observations will be easily understood, when we attempt to perform the operations on a few examples in complex numbers.

Let it be proposed to add together the several sums of money here expressed: —

£	s.	d.
25	17	6
6	13	$5\frac{1}{2}$
4	0	$3\frac{1}{4}$
10	11	$7\frac{3}{4}$
12	14	$6\frac{1}{8}$
7	6	9
	17	$4\frac{1}{2}$
		$7\frac{1}{2}$

To perform this addition, we shall first add together all the farthings which occur after the pence, counting each halfpenny as 2 farthings; the total number of farthings which we shall obtain is 13; but since every 4 farthings is equivalent to 1 penny, 13 farthings will be equivalent to 3 pence and 1 farthing: we therefore write down 1 farthing, and *carry* 3 to the pence. In general, therefore, when the farthings in the sums to be added are added together, we must divide their number by 4, put down the remainder as farthings, and carry the quotient.

We now add the number 3 carried from the farthings to the column of pence, and adding that column we find the total number to be 50 pence; but since every 12 pence makes 1 shilling, we shall find the number of shillings in 50 pence by dividing it by 12: 50 pence, therefore, are 4s. 2d.; we write 2 in the pence place, and carry 4 to the shillings column. In general, therefore, when the pence column is added together with the pence carried from the farthings, we must divide the sum by 12, write the remainder as pence, and carry the quotient to the shillings.

We next add the shillings' column, including the 4 carried from the pence, and we find the sum to be 82: dividing this by 20, we find it equal to 4l. 2s.; we write 2 in the shillings' column, and carry 4 to the pounds. In general, therefore, when the shillings' column is added, we divide by 20, write the remainder in the shillings' place, and carry the quotient to the pounds.

The process of adding the shillings may be facilitated in the following manner:—Add the digits only which occupy their units' places in the first instance; write the figure as you would in simple numbers in the units' place of the shillings of the sum; carry to the tens in the same manner as for simple numbers, and add the tens' column. If the sum be an even number, carry half of it to the pounds, and write no figure in the tens' place of the shillings; but if the sum be an odd number, write 1 in the tens' place of the shillings, and carry half

the remainder to the pounds: thus, if, by adding the units of the shillings column we obtain 75, we write 5 in the units' place of shillings, and carry 7 to the tens; if, after adding the tens with the 7 carried, the sum be 12, we carry 6 to the pounds; but if the sum be 13, we write 1 in the tens' place of the shillings, and carry 6 to the pounds.

By generalising the method adopted in this example, we shall perceive that all complex numbers may be added together by the following

RULE.

(339.) *Place the complex numbers one under another, so that the same classes of units shall stand in the same vertical column, the smallest units occupying the first column on the right, and the units of succeeding orders being placed in successive columns from right to left. Add the first column on the right, and having found its sum, divide it by that number which expresses the number of units of that order contained in a single unit of the next order above it; the quotient will give the number to be carried, and the remainder will give the number to be placed under the first column to the right: add the next column on the left together with the number carried, and having obtained the sum, divide it by that number which expresses how often the unit of the column added is contained in the next superior unit; write the remainder under the column, and carry the quotient to the next column on the left, and proceed in the same way until all the columns of units have been added.*

This rule will be better understood when applied to the following examples. Let it be required to add together the following times:—

years.	weeks	days.	hours.	min.	seconds.
3	27	5	22	51	37
21	15	6	14	32	25
41	49	4	0	0	59
.	38	6	23	47	42
<hr/>					
67	28	2	13	12	43

By adding the seconds column we find the total number 163; this divided by 60 gives the quotient 2, with the remainder 43; we write 43 under the column, and carry 2 to the minutes. The addition of the minutes' column with the 2 carried gives 132; this divided by 60 gives the quotient 2 and a remainder 12: we write 12 under the minutes' column and carry 2 to the hours. Adding the hours with the 2 carried, we find the sum 61: this divided by 24 gives the quotient 2 and the remainder 13: we write 13 in the hours' place and carry 2 to the days. The addition of the column of days gives 23, which, divided by 7, gives the quotient 3 with a remainder 2: writing 2 under the days and carrying 3 to the weeks, we find the total weeks 132; this divided by 52 gives a quotient 2 and a remainder 28: we write 28 under the weeks and carry 2 to the years, by the addition of which we get 67.

The process of adding complex numbers may sometimes be facilitated by omitting the numbers carried in the first instance, and adding each successive column as simple numbers. We shall proceed thus in the following example. Let it be required to add the following distances: —

	furl.	perch.	yds.	ft.	inches.
11	6	29	4	2	11
	7	38	3	0	8
	5	27	0	1	10
	4	35	5	2	9
41	22	129	12	5	38

We have here obtained the true sum of the distances, but the several classes of units are not expressed in their least terms: 38 inches may be reduced to feet and inches by dividing it by 12, and we find that it is equivalent to 3 feet 2 inches; we, therefore, instead of 38 inches, write 2 inches and add 2 to the number of feet. We should thus have 7 feet in the second column; but 7 feet are equivalent to 2 yards 1 foot: we write, therefore, 1 under the column of feet, and carry 2 to the

yards; the number of yards thus becomes 14: but since 11 yards are 2 perches, 14 yards are 2 perches 3 yards; we therefore write 3 under the yards, and carry 2 to the perches: the number of perches thus becomes 131. Dividing this by 40, it will be reduced to furlongs and perches, and is equivalent to 3 furlongs 11 perches: we therefore write 11 under the perches and carry 3 to the furlongs; the number of furlongs will thus be 25: to reduce this to miles and furlongs, we divide it by 8. It is, consequently, equivalent to 3 miles 1 furlong: we therefore write 1 under the furlongs and carry 3 to the miles. The total sum is therefore expressed as follows:—

miles.	furlong.	perches.	yds.	ft.	inches.
44	1	11	3	1	2

(340.) One complex number is subtracted from another by placing the subtrahend under the minuend, the units of the same classes being placed in the same vertical columns. The units of each class of the subtrahend are then subtracted from those of the same class in the minuend, and the results are written under them in the remainder. If the number of units of each class in the subtrahend be less than those of the same class in the minuend, the process will be evident, as in the following examples. Subtract $3l. 7s. 6\frac{1}{2}d.$ from $7l. 10s. 8\frac{1}{4}d.$:—

£	s.	d.
7	10	$8\frac{1}{4}$
3	7	$6\frac{1}{2}$
<hr/>		
4	3	$2\frac{1}{4}$

Subtract 7 cwt. 2 qrs. 8 lbs. 6 oz. from 12 cwt. 3 qrs. 10 lbs. 11 oz.:—

cwt.	qrs.	lbs.	oz.
12	3	10	11
7	2	8	6
<hr/>			
5	1	2	5

If the units of any class in the subtrahend be greater in number than the units of the same class in the minuend, the subtraction cannot be immediately performed:

the difficulty, however, is removed by the same artifice as was explained in the like case in simple numbers. A unit of a *higher* order is carried to the next column, and an equivalent number of units of the *same* order is added to the number in the minuend: thus equal quantities are added to the minuend and subtrahend, and therefore their difference remains the same. (97.) (108.)

Let it be required to subtract 8*l.* 17*s.* 10*d.* from 11*l.* 5*s.* 4*d.*

£	s.	d.
11	5	4
8	17	10
<hr/>		
2	7	6

The number of pence in the minuend being less than the number of pence in the subtrahend, the latter cannot be subtracted from the former: the class of units next superior to pence being shillings, and 1 shilling being equal to 12 pence, we add 12 to the pence in the minuend, by which we obtain 16 pence: subtracting 10 from this, we write the remainder 6 under the pence. To compensate for the 12 pence added to the minuend, we now add 1 shilling to the subtrahend, or *carry* 1 to the shillings: we have then to subtract 18 from 5; but that being impossible, and the class of units next superior being pounds, and 1 pound being equal to 20 shillings, we add 20 shillings to the minuend, by which we obtain 25 shillings: subtracting 18 from this, we write 7 in the remainder. To compensate for the 20 shillings added to the minuend, we add 1 pound to the subtrahend, or *carry* 1 to the pounds: we accordingly subtract 9 from 11, and obtain the remainder 2: the total remainder is then 2*l.* 7*s.* 6*d.*

Let it be required to subtract 3 cwt. 3 qrs. 24 lbs. 12 oz. from 7 cwt. 1 qr. 5 lbs.

cwt.	qrs.	lbs.	oz.
7	1	5	0
3	3	24	12
<hr/>			
3	1	8	4

Since there are no ounces in the minuend, and the unit next superior, 1 pound, being equal to 16 ounces, we subtract 12 ounces from 16, and write 4 in the remainder. To compensate for the 16 ounces added to the minuend, we add 1 pound to the subtrahend; but since 25 is greater than 5, and the unit next superior, 1 quarter, being equal to 28 pounds, we add 28 to the minuend, and subtract 25 from 33, and write 8 in the remainder. To compensate for the 28 pounds added to the minuend, we add 1 quarter to the subtrahend, or carry 1 to the quarters: we have then to subtract 4 from 1; but 1 being less, and the next superior unit, 1 hundred-weight, being equal to 4 quarters, we add 4 to the quarters, and, subtracting 4 from 5, write 1 in the remainder. To compensate for the 4 quarters added to the subtrahend, we add 1 hundred-weight to the minuend, or carry 1 to the hundred-weights; we write 3 in the remainder, and the total remainder is 3 cwt. 1 qr. 8 lbs. 4 oz.

(341.) When these examples are duly considered, the student will perceive, that when the number of units of any class of the subtrahend is greater than the number of units of the same class in the minuend, instead of using the given minuend and subtrahend, we use a different minuend and subtrahend, but of such a nature as will give by subtraction the same remainder: in fact, we add equal quantities to the minuend and subtrahend, by which their difference will remain unchanged. (97.) Thus, in the above numbers, the given minuend and subtrahend in each have been, in fact, changed in the following manner, previous to subtraction:—

£	s.	d.	cwt.	qrs.	lbs.	oz.
11	25	16	7	5	33	16
9	18	10	4	4	25	12
<hr/>						
2	7	6	3	1	8	4
<hr/>						

In the first example, we have added to the minuend and subtrahend the sum of 1*l.* 1*s.*; but we have added it in a different manner to each. In the minuend we have

added 20 to the shillings and 12 to the pence ; and in the subtrahend we have added 1 to the pounds and 1 to the shillings.

In the second example, we have added 1 cwt. 1 qr. 1 lb. to the minuend and subtrahend ; but, as before, we have made this addition in a different manner in each case. In the minuend we have added 4 to the quarters, 28 to the pounds, and 16 to the ounces ; while in the subtrahend we have added 1 to the hundred-weights, 1 to the quarters, and 1 to the pounds.

The student will feel no difficulty in generalising these ideas, so as to apply them to the subtraction of any complex numbers.

(342.) The same methods of verification already explained in the addition and subtraction of simple numbers are applicable likewise to complex numbers. These operations may be used, therefore, to verify each other. In addition, if from the *sum* be subtracted the sum of all the numbers added, except one, that one must be the remainder, if the work be correct ; and, in subtraction, if the remainder be added to the subtrahend, the sum should be the minuend. Also, if the remainder be subtracted from the minuend, the new remainder should be the subtrahend. These consequences are so evident, that it is unnecessary to illustrate them by examples.

CHAP. III.

OF THE MULTIPLICATION OF COMPLEX NUMBERS.

(343.) THE multiplication of complex numbers is an operation which, in many cases, is attended with considerable complexity and difficulty. The methods of performing it will be most clearly understood by practical examples of its application : we shall, therefore, proceed to give a series of examples, ascending from the most simple to the most complex cases ; and shall subjoin to each example such observations as will enable the student to generalise the methods which it suggests.

(344.) FIRST CASE. Let the multiplier be a simple number, not exceeding 12.

Example 1. A piece of a certain cloth costs *2l. 7s. 9d.* ; what will be the price of 7 pieces of the same cloth ? To solve this question, it is necessary to multiply *2l. 7s. 9d.* by 7 : the process is as follows :—

£	s.	d.
2	7	9
<hr/>		
14	49	63
<hr/>		
16	14	3
<hr/>		

We have here multiplied the pounds, shillings, and pence severally by 7, and obtained the three products 14 pounds, 49 shillings, and 63 pence : but since 63 pence admits of being reduced to shillings, and 49 shillings to pounds, we divide 63 by 12, and, writing the remainder, 3, carry the quotient, 5, to the shillings, which makes the number of shillings 54. Dividing this by 20, we write the remainder, 14, in the shillings, and carry the quotient, 2, to the pounds : the product is,

therefore, reduced to the form 16*l.* 14*s.* 3*d.*, which is the answer to the proposed question.

It is not necessary, however, to write the product in the first form, since the reduction may be made at the same time with the process of multiplication: thus, we say, 7 times 9 are 63 pence, which are 5*s.* 3*d.*; we write 3 in the pence, and carry 5 to the shillings: 7 times 7 are 49 shillings, which, with 5 carried, make 54; this is equivalent to 2*l.* 14*s.*; we write 14 in the shillings, and carry 2 to the pounds: 7 times 2 are 14, and 2 carried are 16; we write 16 in the pounds.

Example 2. An engineer contracts to make 12 miles of road, at 367*l.* 15*s.* 4½*d.* a mile; what will be the cost of the whole length? We must multiply the cost of 1 mile by the number of miles:—

£	s.	d.
367	15	4½
		12
<hr/>		
4413	4	6

We say 12 halfpence are 6 pence, carry 6 to the pence: 12 times 4 are 48, and 6 are 54, which is equivalent to 4*s.* 6*d.*; write 6 in the pence, and carry 4 to the shillings: 12 times 5 are 60 and 4 are 64, write 4 in the units' place of the shillings, and carry 6 to the tens: 12 times 1 are 12, and 6 carried are 18; this being 18 ten shillings, is equivalent to 9 twenty shillings, or 9*l.*; carry 9 to the pounds. 12 times 7 are 84, and 9 are 93; write 3 in the units of the pounds and carry 9 to the tens; 12 times 6 are 72 and 9 are 81, write 1 in the tens and carry 8 to the hundreds; 12 times 3 are 36 and 8 are 44, write 4 in the hundreds and 4 in the thousands.

In these examples, it will be perceived that the method adopted is to multiply each class of units in the multiplicand separately by the multiplier, beginning with the inferior classes of units and proceeding to the superior. When the product of each class of units is found, it is reduced to the superior class by dividing it

by the number which expresses how many inferior units are contained in the superior. Thus we find first the product of the farthings or halfpence, if there be any, and to find the number to be carried, reduce that to pence ; we then find the product of the pence, adding the number carried, and reduce that to shillings ; we then find the product of the shillings, adding the number carried, and reduce that to pounds ; finally, we find the product of the pounds, adding the number carried.

This method will always be sufficient for every multiplier which does not exceed the extent of our knowledge of the multiplication table. Since the multiplication table is usually committed to memory for numbers as far as 12, the process of multiplication may commonly be performed in this way, when the multiplier does not exceed 12 ; but the same method will serve for higher numbers with those who have committed to memory the multiplication table to a greater extent.

(345.) **SECOND CASE.** When the multiplier is a number which is the product of two figures neither of which exceeds 12.

Example 1. A certain cloth costs 2*l.* 5*s.* 7*d.* per yard : what is the price of 72 yards ? 72 being the product of 9 and 8, we may multiply the multiplicand first by 9, and then multiply the product by 8 : the product finally obtained will be that which is sought. (140.)

£	s.	d.
2	5	7
		9
<hr/>		
20	10	3
		8
<hr/>		
164	2	0
<hr/>		

Example 2. The weight 3qrs. 17lbs. 11oz. of a certain grain is purchased for 1*l.* ; how much of the same grain can be purchased for 9*l.* ?

96 being the product of 12 and 8, we shall obtain the product sought by first multiplying the multiplicand by 12, and then multiplying the product by 8.

cwt.	qrs.	lbs.	oz.
0	3	17	11
			12
<hr/>			
10	3	15	9
			8
<hr/>			
87	0	12	8

The numbers to be carried from each class of units to the superior class, are determined here in the same manner as already explained in Addition.

(346.) THIRD CASE. When the multiplier is a whole number which cannot be conveniently resolved into small factors.

Example 1. Let it be required to multiply 784*l.* 15*s.* 9*d.* by 857: the process would be performed as follows:—

		£	s.	d.
		784	15	9
		857		
		<hr/>		
		5488		
		3920		
		6272		
s.	d.			
10	6	428	10	0
5	0	214	5	0
0	6	21	8	6
0	3	10	14	3
		<hr/>		
		£672562	17	9

We first multiply the pounds of the multiplicand by the multiplier, according to the rule for the multiplication of whole numbers; but we postpone the addition of the three partial products until the shillings and pence of the multiplicand have been likewise multiplied by the

multiplier. We might proceed to multiply the shillings by the multiplier by the rules for whole numbers, and obtain the products, which in that case would be expressed in shillings, and might be subsequently reduced to pounds; but we obtain the result by a more abridged though less direct process: we consider the 15 shillings of the multiplicand to be resolved into two parts, viz. 10 shillings, and 5 shillings: 10 shillings being the half of a pound, we should obtain by multiplying it by the multiplier half as many pounds as are expressed by that multiplier; we have, therefore, only to consider the multiplier as expressing pounds, and to divide it by 2, in order to obtain the product of 10 shillings multiplied by it. Dividing 857 by 2, we obtain the quotient 428, with a remainder 1, which expresses half a pound, or 10 shillings: the product, therefore, of 10 shillings by 857 is 428*l.* 10*s.* It now remains to multiply 5 shillings by the multiplier, but this product will evidently be half the preceding product in which the multiplicand was 10 shillings: dividing the preceding product by 2, we obtain 214*l.* 5*s.*, which is, therefore, the product of 5 shillings by the multiplier. It now remains to multiply 9 pence by the multiplier, and we accomplish this by considering 9 pence to be resolved into two parts, 6 pence, and 3 pence: 6 pence being the tenth part of 5 shillings, we shall obtain the product of 6 pence by the multiplier by taking the tenth part of the preceding product; dividing the pounds by 10, we get 21 with a remainder 4; this, being 4 pounds, is equivalent to 80 shillings, to which the 5 shillings being added, we obtain 85 shillings; this, divided by 10, gives the product 8, with a remainder 5; this 5 shillings is equivalent to 60 pence, which being divided by 10 gives the quotient 6: the partial product is therefore 21*l.* 8*s.* 6*d.* The product of 3 pence by the multiplier is, evidently, half the last found product; to obtain it, therefore, we have only to divide the last product by 2: dividing the pounds by 2, we get the quotient 10, with a remainder 1: this

1 pound being equivalent to 20 shillings, and added to the 8 shillings, we divide 28 by 2, and get the quotient 14. We next divide 6 by 2, and get the quotient 3 : the last partial product is therefore 10*l.* 14*s.* 3*d.* All these partial products being now added together, we obtain the total product.

By the method here pursued, we multiply, in the first instance, the highest class of units in the multiplicand by the multiplier, following the rules established for simple numbers : the shillings are then resolved into parts, one or more of which are aliquot parts, or submultiples of a pound. In the present case, the number being 15, we resolve it into 2 parts, one of which, 10, is the half of a pound, and the other is the half of that, or the fourth of a pound : had the number of shillings been 17, we should have resolved it into 10, 5, and 2, the first being half of a pound, the second half the first, and the third a fifth of the first. Having found the first partial product, its half would be the second, and its fifth the third.

The pence are resolved into such parts that one of them shall be an aliquot part of one of the parts into which the shillings have been resolved. In the present case, 6 pence is the tenth of 5 shillings, and we, accordingly, find the product corresponding to 6 pence by dividing the product corresponding to 5 shillings by 10. Had the number of shillings been 17, we should have had a product corresponding to 2 shillings, in which case we should have found the product corresponding to 6 pence by dividing the latter by 4.

The spirit of this method consists in resolving the shillings and pence into a series of parts, each of which shall be an aliquot part or submultiple of some preceding part, so that the successive partial products may be derived one from another by dividing by single digits.

Example 2. Let it be required to multiply 67 fathoms, 5 feet, 6 inches, 5 lines, by 59 (1 fathom = 6 feet) : the process will be as follows : —

	fath.	ft.	ins.	lines.
	67	5	6	5
	59			
	603			
	335			
3 feet.....	29	3		
2 feet.....	19	4		
6 inches.....	4	5	6	
1 inch	0	4	11	
4 lines	0	1	7	8
1 line.....	0	0	4	11
	4007	2	6	7

As before, we begin by multiplying the highest class of units by the multiplier, as in whole numbers, and obtain the first two partial products, which are arranged according to the rule for simple multiplication. The partial products corresponding to the inferior classes of units are next found : 3 feet being half a fathom, we resolve the 5 feet of the multiplicand into 3 and 2 ; we may obtain the partial product corresponding to 3 feet by dividing the multiplier by 2 : the quotient is 29, with a remainder 1 : this being a fathom, when divided by 2 we obtain the quotient 3 feet. The partial product is therefore 29 fathoms 3 feet : 2 feet being the third part of a fathom, we obtain the fourth partial product by dividing 59 fathoms by 3 ; the quotient is 19, with a remainder 2, which, divided by 3, gives two thirds of a fathom, or 4 feet : 6 inches being the fourth part of 2 feet, we obtain the fifth partial product by dividing the fourth by 4 : 19 fathoms divided by 4 gives the quotient 4, with a remainder 3 fathoms, or 18 feet : this added to 4 feet gives 22 feet, which divided by 4 gives the quotient 5, with a remainder 2 feet, or 24 inches : this divided by 4 gives the quotient 6 inches. The fifth partial product is, therefore, 4 fathoms 5 feet 6 inches. To facilitate the remaining partial products, we first obtain that which would correspond to 1 inch, which is done by dividing the fifth partial product by 6 : 4 fathoms 5 feet being equivalent to 29 feet, when divided by 6 gives the quotient 4 feet, with a remainder 5 feet, or 60 inches ; this added to 6 inches gives 66 inches,

which, divided by 6, gives the quotient 11. This number, 4 feet 11 inches, would be the partial product, therefore, corresponding to 1 inch, or 12 lines. Let the 5 lines in the multiplicand be supposed to be resolved into 4 and 1: the partial product corresponding to 4 lines will be found by dividing 4 feet 11 inches by 3, since 4 is the third part of 12: dividing 4 by 3, we get the quotient 1, with a remainder 1 foot, or 12 inches, to which 11 being added, we get 23 inches; which, divided by 3, gives the quotient 7, with the remainder 2 inches, or 24 lines. Dividing 24 lines by 3, we get the quotient 8 lines: the sixth partial product is, therefore, 1 foot 7 inches 8 lines; the seventh partial product, corresponding to 1 line, will be the fourth part of this: dividing, therefore, 1 foot 7 inches, or 19 inches, by 4, we get the quotient 4, with a remainder 3 inches, or 36 lines: this, added to 8 lines, gives 44 lines, which, divided by 4, gives the quotient 11 lines. The last partial quotient is, therefore, 4 inches 11 lines.

The product corresponding to 1 inch being introduced merely for the purpose of facilitating the process by which the remaining partial quotients are found, and not forming a part of the actual multiplication, is omitted in the addition, and, to indicate this, lines have been drawn across the figures.

(347.) When the multiplicand is a sum of money, the process of multiplication may frequently be simplified by converting it into decimal parts of a pound, by the method already explained (336. *et seq.*).

Example 1. Let it be required to find the annual amount of 3*l.* 13*s.* 6*d.* per day: 3*l.* 13*s.* 6*d.* = 3.675. We must therefore multiply the latter number by 365, the number of days in the year: the process will be as follows: —

$$\begin{array}{r}
 3.675 \\
 365 \\
 \hline
 18375 \\
 22050 \\
 11025 \\
 \hline
 \end{array}$$

£1341.375 = £1341 7 6

The decimal found by this process of multiplication is here reconverted into pounds, shillings, and pence, by the method explained in (337).

Example 2. The wages of an artisan are 2*l.* 10*s.* 6*d.* per week; what are his annual wages? The weekly wages must here be multiplied by 52, the number of weeks in a year. The process is as follows:—

$$\begin{array}{r}
 \text{£} \quad \text{s.} \quad \text{d.} \\
 2 \quad 10 \quad 6 = 2.525 \\
 52 \\
 5050 \\
 12625 \\
 \hline
 \text{£}131.300 = \text{£}131 \quad 5 \quad 0
 \end{array}$$

(348.) When the multiplicand is a sum of money, the process of multiplication is frequently facilitated by resolving the shillings and pence into sub-multiples, or aliquot parts, of a pound. To do this with facility, it would be necessary that the computer should commit to memory the principal subdivisions of the pound sterling, which are expressed in the following table:—

£	d.	s.	d.
$\frac{1}{2}$ =	0	$\frac{1}{2}$ =	1 0
$\frac{1}{3}$ =	8	$\frac{1}{4}$ =	0 10
$\frac{1}{4}$ =	0	$\frac{1}{5}$ =	0 8
$\frac{1}{5}$ =	0	$\frac{1}{6}$ =	0 7½
$\frac{1}{8}$ =	4	$\frac{1}{10}$ =	0 6
$\frac{1}{10}$ =	6		0 5
$\frac{1}{12}$ =	0		0 4
$\frac{1}{15}$ =	8	$\frac{1}{84}$ =	0 3½
$\frac{1}{18}$ =	4		
$\frac{1}{20}$ =	3		

It will be also advantageous to commit to memory the following table of aliquot parts of a shilling:—

s.	d.	s.	d.
$\frac{1}{2}$ =	6	$\frac{1}{12}$ =	1
$\frac{1}{3}$ =	4	$\frac{1}{18}$ =	½
$\frac{1}{4}$ =	3	$\frac{1}{24}$ =	½
$\frac{1}{6}$ =	2	$\frac{1}{36}$ =	¼
$\frac{1}{8}$ =	1½		

The use of these tables will be perceived in the following examples.

Example 1. The wages of a labourer are 1*l.* 7*s.* 8*d.* per week ; what will be his yearly receipt ? It is necessary to multiply the weekly wages by 52 ; the process is as follows : —

				£	s.	d.
				1	7	8
			52			
			<hr/>			
s.	d.		52			
6	8	= $\frac{1}{3}$	17	6	8
1	0	= $\frac{1}{20}$	2	12	0
				£71	18	8

We first multiply the pounds by 52, and obtain the first partial product ; we then resolve 7*s.* 8*d.* into 6*s.* 8*d.*, and 1*s.*, which are respectively $\frac{1}{3}$ and $\frac{1}{20}$ of a pound. Since 1*l.* a week would be 52*l.* a year, $\frac{1}{3}$ and $\frac{1}{20}$ of a pound will be respectively $\frac{1}{3}$ and $\frac{1}{20}$ of 52*l.* a year : we therefore divide 52*l.* by 3 and by 20, and we obtain the second and third partial products, which are, in fact, the products of 6*s.* 8*d.* and 1*s.* multiplied by 52.

Example 2. What is the annual amount of 17*s.* 11*d.* per day ? It is necessary to multiply this sum by 365 : to do so, we shall resolve it in such a manner that the first part shall be an aliquot part of a pound, and each succeeding part an aliquot part of the preceding one. If we take 10 shillings for the first part, 5 shillings for the second, 2 shillings and 6 pence for the third, and 5 pence for the fourth, this will be accomplished. The first will then be half of a pound, the second half the first, the third half the second, and the fourth the sixth part of the third, and the four partial products will have corresponding relations. We shall find the amount of 10*s.* a day by dividing 365*l.* by 2 ; the quotient is 182*l.* 10*s.* : the remaining partial products are found as follows : —

s.	d.		£	s.	d.
10	0	per day =	182	10	0 per annum.
5	0	— =	91	5	0 —
2	6	— =	45	12	6 —
0	5	— =	7	12	1 —
			<hr/> £326 19 7 <hr/>		

In this case, the second partial product is found by dividing the first by 2; the third by dividing the second by 2; and the fourth, by dividing the third by 6.

(349.) This method of multiplication, by resolving the multiplicand into a number of parts, which are either sub-multiples of each other, or sub-multiples of one of the principal units of the multiplicand, is usually delivered, in treatises on Arithmetic, as a distinct rule, under the name of *PRACTICE*. It is, however, as will be easily perceived, nothing but a peculiar method of multiplication, and the facility with which it may be performed will depend on the expertness and ingenuity with which complex numbers may be resolved into parts having the peculiar relation to each other just mentioned. No general rule can be given for such a resolution, and there are many ways in which the same number may be thus resolved. Thus, the above sum might have been resolved into 10s., 6s. 8d., and 1s. 3d., in which case the three partial products would have been found by dividing 365 by 2, 3, and 16 successively: the process would be as follows:—

s.	d.		£	s.	d.
10	0	per day =	182	10	0 per annum.
6	8	— =	121	13	4 —
1	3	— =	22	16	3 —
			<hr/> £326 19 7 <hr/>		

In this case, the parts are all sub-multiples of the principal unit of the multiplicand; but the process is not as concise, and is more liable to error in calculation than when the parts are taken so as to be sub-multiples of each other.

Example. What is the price of 165 pieces of silk at 7*l.* 13*s.* 7½*d.*: we shall resolve this sum into the following parts, 7*l.*, 10*s.*, 3*s.* 4*d.*, 3*d.*, and ½*d.*: the process will be as follows:—

£	s.	d.		£	s.	d.
7	0	0	1155	0	0
0	10	0	82	10	0
0	3	4	27	10	0
0	1	0	8	8	8
0	0	3	2	1	3
0	0	0½	0	6	10½
				<hr/>		
				£1267	8	1½

The first product is the price of 165 yards at 7*l.* per yard; the second at 10*s.* per yard is found by dividing 165 by 2, 10*s.* being half of a pound; the third is found by dividing the second by 3, for 3*s.* 4*d.* is the third part of 10*s.* In order to facilitate the discovery of the other partial products, we now find the product which would correspond to 1*s.* 0*d.*, which is done by dividing the product corresponding to 10*s.* by 10: the product corresponding to 1*s.* being thus found, we find the product corresponding to 3*d.* by dividing it by 4; and, as the former product does not constitute a part of the question, but is merely introduced to facilitate the calculation, we strike it out. The last partial product corresponding to ½*d.* is found by dividing the preceding product by 6, since ½*d.* is the sixth part of 3*d.*

The same question solved by the decimal method explained in (336.), would be as follows:— 7*l.* 13*s.* 7½*d.* = 7.6817. Multiplying this by 165,—

$$\begin{array}{r}
 7.681 \\
 165 \\
 \hline
 38405 \\
 46086 \\
 681 \\
 \hline
 \end{array}$$

$$\underline{\underline{£1267.365}} = \underline{\underline{£1267 \quad 7 \quad 4\frac{1}{2}}}$$

It will be observed that this result is less by $9\frac{1}{4}d.$ than the sum found by the other method: this has not resulted from any error in the work, but has arisen from another circumstance, which admits of easy explanation. It will be recollected that the method of finding the equivalent decimal for any sum of money, explained in (336.), is accurate only as far as the third place of decimals, which is all that is necessary when it is not required to express fractions less than a farthing; but when the sum is to be multiplied by any high number, as in the present case, where the multiplier is 165, the figures which would fill the places beyond the third place of decimals would, after the multiplication, affect the second and third places, because numbers would then be carried to these places. In the present instance, the decimal places, instead of being $\cdot 365$, would have been $\cdot 406$: the error is, however, trifling in amount; and, therefore, where extreme accuracy is not sought, the rule may still be applied.

(350.) In ordinary cases, it seldom happens that the sums which fall under the hands of the computer include parts of a shilling less than $6d.$, especially where they are prices of goods sold per piece, or of pay per week or day. If the sum to be computed have $6d.$ in the pence, then the method just referred to will still give an accurate result, however high the multiplier may be. We shall apply both methods to the following example:—

Example.—A certain stuff costs $3l. 13s. 6d.$ a piece; what is the price of 376 pieces?

We shall resolve the given sum into the following parts $3l.$, $12s.$, and $1s. 6d.$: the calculation will be as follows:—

£	s.	d.		£	s.	d.
3	0	0	1128	0	0
0	12	0	225	12	0
0	1	6	28	4	0
				<hr/>		
				£1381 16 0		

The first product is found by multiplying 376 by 3, since 12 shillings is the fifth part of 3*l.*; the second product is found by dividing the first by 5; and since 1*s.* 6*d.* is the eighth part of 12*s.*, the third product is found by dividing the second by 8.

The same question will be solved by the decimal method as follows:—

$$\begin{array}{r} \text{£} \quad \text{s.} \quad \text{d.} \\ 3 \quad 13 \quad 6 \quad \cdot \quad \text{£}3\cdot675 \end{array}$$

Multiply this by 376:—

$$\begin{array}{r} 3\cdot675 \\ 376 \\ \hline 22050 \\ 25725 \\ 11025 \\ \hline \text{£}1381\cdot800 = \text{£}1381 \quad 16 \quad 0 \end{array}$$

(351.) **FOURTH CASE.** When the multiplier is a fraction, we have already explained that, to multiply any quantity by a fraction, it is only necessary to multiply that quantity by its numerator, and to divide the product by its denominator. Thus, to multiply by a fraction involves, in fact, the multiplication and division by whole numbers. When the denominator of the fraction does not exceed 12, the process is sufficiently simple, and has been applied in the preceding examples; but if the denominator be a large number, the case must be reserved for investigation in the following chapter.

If the numerator of the fraction be 1, the multiplication is performed by merely dividing by the denominator: in fact, multiplying by such a fraction is equivalent to division. Thus, to multiply by $\frac{1}{10}$, and to divide by 10, are one and the same operation. In practice, it is frequently convenient when a fraction occurs having a numerator greater than 1, to resolve it into several fractions, whose numerator shall be 1, and

to obtain several partial products, or rather quotients, by dividing the multiplicand by their denominators severally: thus, if the multiplier were $\frac{9}{10}$, we should resolve it into three fractions, $\frac{3}{10}$, $\frac{2}{10}$, and $\frac{4}{10}$, or, what is the same, $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{1}{5}$. We should in this case obtain the three partial products by dividing the multiplicand, first by 2, then by 5, and again by 5. Again, if the multiplier was $\frac{1}{2}$, we should resolve it into $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{4}$; or $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{4}$. We should obtain the first partial product, by dividing the multiplicand by 2; and since $\frac{1}{2}$ is the half of $\frac{1}{2}$, we should obtain the second partial product by dividing the first by 2: and since $\frac{1}{4}$ is the third of $\frac{1}{2}$, we should obtain the third partial product by dividing the first by 3.

(352.) FIFTH CASE. When the multiplier is a mixed number.

In this case, we must multiply first by the integral part of the multiplier, and then multiply by the fractional part, according to the method explained in the preceding case: the two products being added together, the total product will be found.

Example. Let it be required to find the total amount of 65*l.* 17*s.* 11*d.* per annum for $39\frac{7}{8}$ years: the process is as follows:—

		£	s.	d.
		65	17	11
		<u>397</u>		
		585		
		<u>195</u>		
10	0	19	10	0
5	0	9	15	0
2	0	3	18	0
0	6	0	19	6
0	3	0	9	9
0	2	0	6	6
0	4	32	18	11½
0	16	9	5½
0	8	4	8½
		<u>£2627 11 11½</u>		

As in the former ~~examples~~, we first multiply the pounds by 39; we then resolve 17*s.* into 10*s.*, 5*s.*, and 2*s.* : to multiply 10*s.* by 39, we consider 39 as pounds, and divide it by 2, and obtain the third partial product; to find the next, we divide this latter by 2, since 5 is the half of 10; and, to find the following one, we divide it by 5. Since 2 is the fifth of 10, we resolve 11*d.* into 6*d.*, 3*d.*, and 2*d.* : we find the product corresponding to 6*d.* by dividing the product corresponding to 5 shillings by 10; we find the product corresponding to 3*d.* by dividing that corresponding to 6*d.* by 2, and we find the product corresponding to 2*d.* by dividing it by 3. We have thus obtained all the partial products necessary to compose the product of the multiplicand by 39: but it still remains to multiply the multiplicand by the fraction $\frac{7}{8}$: to accomplish this, we have resolved the fraction into three parts, viz. $\frac{1}{8}$, $\frac{2}{8}$, and $\frac{1}{8}$: since $\frac{1}{8}$ is $\frac{1}{2}$, to find the product corresponding to it, we have only to divide the multiplicand by 2, by which we obtain the corresponding partial product: to find the product corresponding to $\frac{2}{8}$, which is the half of $\frac{1}{8}$, we have only to divide the last product by 2. By again dividing the product thus found by 2, we find the product corresponding to $\frac{1}{8}$. The addition of all these sums gives the total product sought.

The same question might be solved by the decimal method in the following manner:—Let us first find the value of 65*l.* 18*s.* 0*d.* per annum for $39\frac{7}{8}$ years, and then subtract from the result the amount of a penny a year for $39\frac{7}{8}$ years; that is, subtract the $39\frac{7}{8}$ pence: the process will be as follows:—

$$£65\ 18\ 0 = £65.9.$$

We shall first find the product corresponding to 39, and shall afterwards obtain the product corresponding to $\frac{7}{8}$.

659	8)65·9000
39	<u>8·2875</u>
5931	7
1977	<u>57·6625</u>
<u>25701</u>	

39 years	2570·1
$\frac{1}{2}$ —	57·6625
	<u>2627·7625 = £2627 15 3</u>
Subtract $39\frac{1}{2}$	3 $3\frac{1}{2}$
	<u>£2627 11 11$\frac{1}{4}$</u>

(353.) The artifice used here for the simplification of the process is one of which we may frequently avail ourselves with advantage. It consists in taking a multiplicand somewhat greater than the one proposed ; but which, being expressed more in whole numbers, renders the process of multiplication more expeditious. The excess is afterwards compensated for by subtraction. The same artifice may be always used in all cases where the multiplicand is a little less than a round number.

Example. Let it be required to find the total amount of 9*l.* 19*s.* 11*d.* payable yearly for 100 years.

Had the sum been 10*l.* the amount would evidently be 1000*l.* ; but the sum proposed is 1*d.* less than 10*l.*, and therefore the sum sought will be 100 pence less than 1000*l.* We shall find the sum sought by subtracting 8*s.* 4*d.* from 1000*l.* : the result is 999*l.* 11*s.* 8*d.*

(354.) **SIXTH CASE.** When the multiplier is a compound number.

Example 1. What is the expense of repairing a road, the length of which is 69 miles, 6 furlongs, 25 perches, at 25*l.* 19*s.* 5*d.* per mile ?

We shall first ascertain the price for 69 miles, which will be done by multiplying 25*l.* 19*s.* 5*d.* by 69, by the methods explained in the preceding cases : we must next resolve the furlongs and perches into sub-multiples, or aliquot parts, of a mile or of each other : we shall resolve the furlongs into 4 furlongs, which is half a mile

and 2 furlongs, which is half the latter. We shall resolve the perches into 20 perches, which is half a furlong, or the fourth part of 2 furlongs, and 5 perches, which is the fourth part of 20 : the process will be as follows : —

			£25 19 5		
			mil. fur. per.		
			69 6 25		
			<hr/>		
			225		
			150		
s.	d.		34	10	0
10	0	-	17	5	0
5	0	-	6	18	0
2	0	-	6	18	0
2	0	-	1	3	0
0	4	-	0	5	9
0	1	-	12	19	8
4 furlongs		-	6	9	10
2	—	-	1	12	5½
20 perches		-	0	8	1¾
5	—	-	<hr/>		
			£1813 9 9¼		

The first 8 products above are the partial products found by multiplying the multiplicand by 69; by their addition we should obtain the cost for 69 miles : the remaining 4 products are the cost for the furlongs and perches in the multiplier ; the first is the cost for 4 furlongs, or half a mile, and is found by taking half the multiplicand ; the next product is half this, being the cost of 2 furlongs ; the following one, the fourth of the latter, being the cost of half a furlong ; or 20 perches ; and the last product is the fourth part of this sum, being the cost of 5 perches. These all added together give the total cost sought.

(355.) In the multiplication of simple numbers, and even in complex numbers, when one of the factors is a simple number, the multiplier and multiplicand may interchange places without producing any change in the product : this has been fully explained with respect to simple numbers in the former book, and due attention to the examples given in the present chapter will show

its truth when one of the factors is a complex and the other a simple number. In this last case the effect of the multiplication will be to add together the complex number, repeated as often as there are units in the simple number, if the latter be a whole number: if it be a mixed number, or a fraction, then the operation will be equivalent to taking as much of it as is proportionate to the value of the fraction (241.). In every such case, it is evident that the product will be a complex number of the same kind as the complex factor: thus, if the complex factor express money, the product will also express money; if the complex number express weight, the product will also express weight, and so on.

When both factors, however, are complex, they lose their quality of being interchangeable, and the multiplier becomes essentially distinct from the multiplicand, and must be regarded in a totally different sense. In the example just given, the object of the operation is, first to repeat the given sum of money as often as there are miles in the proposed distance, then to add the same fractional parts of that sum of money as the furlongs and perches in the given distance are of 1 mile; we might then have proceeded by first multiplying, as above, the sum of money by 69: since 6 furlongs are $\frac{3}{4}$ of a mile, we might then have added to the result $\frac{3}{4}$ of the proposed sum; and, again, since 25 perches are $\frac{5}{8}$ of a furlong, and therefore $\frac{5}{32}$ of a mile, we might have annexed $\frac{5}{32}$ of the cost of 1 mile. The result of the calculation would have been the same, but the process more complex.

It will then be perceived that in this process the principal units of the multiplier are used as a whole number would be if the multiplier was simple, and the units of inferior orders are used merely as fractional parts of the principal units. By this means the multiplier, though complex, is implicitly reduced to a simple number, and the process is the same as if we expunged all the classes of units inferior to the principal units, and substituted in their places the equivalent fractions of the principal unit. In the above number the

multiplier is 69 miles, 6 furlongs, 25 perches. This might be converted into a simple number by expressing it $69\frac{6}{8} \dots \frac{25}{160}$, or, what is the same, $69\frac{3}{4} \dots \frac{5}{32}$.

Let us now consider what the meaning of the process would be, if the places of the multiplicand and multiplier were interchanged. In that case, the effect of the operation would be, in the first place, to repeat the given distance 25 times, and we should accordingly find the distance which would cost 25*l.* if the given distance cost 1*l.* We should next take the same fraction of the given distance as 19*s.* is of a pound, which would be the distance, the repair of which would cost 19*s.*, the whole distance being supposed to cost a pound. In the same manner, we should next take the same fraction of the given distance as 5 pence is of a pound, and we should, as before, find the distance which would cost 5 pence, the whole given distance being supposed to cost 1 pound. The question, therefore, to which such an operation would give the answer would be the following. If the repair of 69 miles, 6 furlongs, 25 perches cost 1*l.*, what distance can be repaired for 25*l.* 19*s.* 5*d.* The result of the operation will in this case be expressed in units similar to those of the multiplicand, viz. miles, furlongs, &c. The actual process would be as follows:—

	mil.	fur.	per.
	69	6	25
	£	s.	d.
	25	19	5
	<hr/>		
	345		
	138		
4 furlongs	-	12	4 0
2 —	-	6	2 0
20 perches	-	1	4 20
5 —	-	0	3 5
10 <i>s.</i> 0 <i>d.</i>	-	34	7 12 $\frac{1}{2}$
5 0	-	17	3 26 $\frac{1}{4}$
2 0	-	6	7 34 $\frac{1}{2}$
2 0	-	6	7 34 $\frac{1}{2}$
0 4	-	1	1 12 $\frac{5}{8}$
0 1	-	0	2 13 $\frac{5}{16}$
	<hr/>		
	1813	3	38 $\frac{13}{16}$

(356.) It is sometimes more convenient and expedient to convert the multiplier into a simple number. In the example just given, we might proceed by first taking 26*l.* as a multiplier, and then subtracting from the result the product which would correspond to 7 pence, the excess of the assumed multiplier above the given one. Proceeding by this method, the process would be as follows:—

	mil.	fur.	perc.		mil.	fur.	perc.
	69	6	25		69	6	25
	26	0	0				
	414	0	0	1 <i>s.</i> Od.	8	8	87½
	138	0	0	6 <i>d.</i>	1	5	38½
4 furlongs -	13	0	0	1 <i>d.</i>	0	2	13½
2 — -	6	4	0		2	0	11½
20 perches -	1	5	0				
5 — -	0	3	10				
	1815	4	10				
Subtract 7 <i>d.</i> -	2	0	11½				
	1813	3	38½				

We have here found the product corresponding to the multiplier 26, by the method explained in (346). To facilitate the determination of the product corresponding to 7 pence, we first find the product corresponding to 1*s.*, which is done by dividing the multiplicand by 20. Half this product gives the product corresponding to 6*d.*, and the sixth part of the latter is the product corresponding to 1*d.*; adding the last two, we obtain the product corresponding to 7*d.*; this subtracted from the product corresponding to 26*l.*, gives the product corresponding to 25*l.* 19*s.* 5*d.*, which is sought.

(357.) The results of the multiplication of complex numbers may be verified, as in simple numbers, by dividing the product by the multiplier. The quotient should in that case be the multiplicand; but it is generally more expeditious to verify the work by working

it twice by different methods, which may always be done by resolving one or both factors into a different series of aliquot parts.

We may also verify by doubling one factor and halving the other. The product, after this change, should remain the same ; or, if one factor only be doubled, we should get a product equal to double the original product.

CHAP. IV.

OF THE DIVISION OF COMPLEX NUMBERS.

IN the same manner as in the last chapter, we shall consider successively the different cases of division, taking the most simple first in order.

(358.) FIRST CASE. When the divisor is a whole number.

Example 1. The sum of 25469*l.* 19*s.* 11*d.* is paid for 568 pipes of wine; what is the cost per pipe?

This question will be solved by dividing the sum proposed into 568 equal parts: one of these parts will be the price of a single pipe. The dividend, therefore, is the sum proposed, and the divisor the number of pipes; the quotient will be the price of one pipe. The process is as follows: —

	<i>£</i>	<i>s.</i>	<i>d.</i>	<i>£</i>	<i>s.</i>	<i>d.</i>
	568)25469	19	11	(44	16	9½
	2272					
	2749					
	2272					
	477	pounds				
Multiply	20					
	9540	shillings				
Add	-	19				
	568)9559(16					
	568					
	3879					
	3408					
	471	shillings				
Multiply	12					
	5652	pence				

$$\begin{array}{r}
 \text{Brought forward} \quad 5652 \\
 \text{Add} \quad - \quad 11 \\
 \hline
 568 \overline{)5663(9} \\
 \quad 5112 \\
 \hline
 \quad \quad 551 \text{ pence} \\
 \text{Multiply} \quad \quad 4 \\
 \hline
 \quad 568 \overline{)2204 \text{ farthings (3}} \\
 \quad \quad 1704 \\
 \hline
 \quad \quad \quad 500
 \end{array}$$

We first divide the pounds by 568, in the same manner as in simple numbers, and find the quotient 44, with a remainder 447. This, being less than the divisor, does not admit of a further division. In order to continue the process, it is converted into shillings, the equivalent number of which is 9540. The number of shillings in the dividend being added to this, the sum which remains to be divided by the divisor is 9559 shillings. This number of shillings being now taken as dividend, we obtain the quotient 16, with a remainder 471 shillings. This latter number, being less than the divisor, is converted into pence, the equivalent number of which is 5652. The pence of the dividend being added to this, we obtain 5663 pence, which is the sum which remains to be divided by the divisor. This being taken as dividend, we obtain the quotient 9 pence, and the remainder 551 pence. The last is reduced to farthings by multiplying by 4, by which we obtain 2204 farthings. This, divided by the divisor, gives the quotient 3, with a remainder 500. The whole quotient, therefore, is 44*l.* 16*s.* 9 $\frac{3}{4}$ *d.*, with a remainder 500, which divided by the divisor would give the fraction of a farthing $\frac{500}{568}$. This fraction, being very little less than 1, is nearly equal to 1 farthing. Thus the quotient 44*l.* 16*s.* 10*d.* is in excess of the true quotient by a very minute fraction of a farthing.

If it be required to express the quotient with the exact fraction of a penny, we might stop the process of division after the quotient 9 had been obtained in the pence. The remainder obtained at that point was 551 pence; this divided by 568 gives the fraction of a penny $\frac{551}{568}$; and the exact quotient is, therefore, 44*l.* 16*s.* 9 $\frac{551}{568}$ *d.*

(359.) The method adopted in this example is, first to divide the principal units of the dividend by the divisor, then to reduce the remainder to units of the order next inferior, and add to it the units of the same order in the dividend. Taking the number thus obtained as dividend, the quotient will be the units of the next inferior order in the quotient sought; and the remainder is reduced, as before, to the units of the next order. The units of that order in the dividend are added to it, and the sum taken is the next partial dividend. The same process would be continued until every class of units of which the complex number is susceptible has been successively obtained by reduction. If there still is found a remainder, then, by taking that remainder as numerator, and the divisor as denominator, we obtain the fraction of the last unit, which is necessary to complete the quotient. These observations will be further illustrated in the following example: —

Example. The sum of 765*l.* is paid for 47 tons, 12 cwt. 3 qrs. 57 lbs. 12 oz. of a certain commodity; what is the quantity which could be obtained for 1*l.*?

To solve this question, it is necessary to divide the proposed weight into as many equal parts as there are pounds in the given sum of money, and one of these parts will be the quantity which may be obtained for 1*l.* We must, therefore, take the proposed weight as dividend, and 765 as divisor, and the quotient will be the weight sought.

	tons.	cwt.	qrs.	lbs.	oz.	cwt.	lbs.	oz.
765)	47	12	3	57	12	(1	27	9 $\frac{7}{16}$
	20							
	940							
	12							
	952							
	765							
	187	cwt.						
	4							
	748							
	3							
	751							
	28							
	6008							
	1502							
	21028	lbs.						
	57							
	21085							
	1530							
	5785							
	5355							
	430							
	16							
	2580							
	430							
	6880	oz.						
	12							
	6892							
	6885							
	7							

The number of tons, the principal units of the dividend being less than the divisor,* does not admit of division ; we therefore convert it to hundred-weights, by multiplying by 20, and add to the product the hundred-weights of the dividend. Taking the number of

hundred-weights thus obtained as the first dividend, we obtain the quotient 1 and the remainder 187. This remainder is reduced to quarters, and the quarters of the dividend added, by which we obtain 751 quarters; but, this being less than the divisor, we convert it into pounds, and add the pounds of the dividend. We thus obtain 21085 pounds, which, being taken as a dividend, we obtain 27 for the pounds of the quotient, with a remainder 430 pounds. This remainder is converted into ounces, and the ounces of the dividend added, which gives 6892 ounces for the next partial dividend. We then obtain 9 for the ounces of the quotient, with a remainder 7. This remainder being taken as numerator, and the divisor as denominator, we get the fraction of an ounce, $\frac{7}{65}$, which completes the division.

(360.) SECOND CASE. When the divisor is a fraction. (This case embraces that in which the divisor is a mixed number, for such a number may always be reduced to an equivalent fraction.)

It has been explained that, in order to divide any quantity by a fraction, it is necessary to divide it by the numerator, and multiply the quotient by the denominator of the fraction. Thus, the present case may always be performed by combining the method explained in the first case with that explained in the first case of the preceding chapter.

Example. If $25\frac{1}{3}$ pieces of a certain cloth cost 41*l.* 13*s.* 7*d.*, what will be the price of one piece? In this case it is necessary to find such a sum as, if repeated 25 times, and to the result the fractional part expressed by $\frac{1}{3}$ of the same sum added, the total would be 41*l.* 13*s.* 7*d.* This will be evidently obtained by dividing the sum of money by $25\frac{1}{3}$.

$$25\frac{1}{3} = \frac{76}{3}.$$

We must, therefore, multiply 41*l.* 13*s.* 7*d.* by 31, and divide the quotient by 796.

			£	s.	d.
			417	18	7
			31		
			<u>417</u>		
s.	d.	1251			
10	0	-	15	10	0
2	0	-	3	2	0
1	0	-	1	11	0
0	6	-	0	15	6
0	1	-	0	2	7
<hr/>					
796)	12948	1	1	(16 5 3 ⁷⁴ ₉₈
		796			
		<u>4988</u>			
		4776			
		<u>212</u>			
		20			
		<u>4240</u>			
		1			
		<u>4241</u>			
		3980			
		<u>261</u>			
		12			
		<u>3132</u>			
		1			
		<u>3133</u>			
		2388			
		<u>745</u>			

(361.) **THIRD CASE.** When the divisor is a complex number of the same kind as the dividend.

Example. To cut a certain canal costs at the rate of 47*l.* 19*s.* 5*d.* per perch; how many perches can be cut for 2728*l.* 17*s.* 10*d.*?

It is evident that the number of perches and parts of a perch sought is the same as the number of times and parts of a time that 47*l.* 19*s.* 5*d.* are contained in 2728*l.* 17*s.* 10*d.*: we must, therefore, divide the latter number by the former, and the quotient will express the number of perches and parts of a perch sought. To effect this division, it is only necessary to reduce the

divisor and dividend to units of the same kind, and then divide the one by the other as simple numbers: thus, if both sums of money be converted into pence, it is only necessary to find how often the lesser number of pence is contained in the greater, which may be done by the rule for the division of whole numbers. The process is as follows: —

£	s.	d.	£	s.	d.
47	19	5	2728	17	10
20			20		
			<hr/>		
959			54577		
12			12		
<hr/>			<hr/>		
11513			11513) 654934	per. yds. ft. inch.	
			57565	(56 4 2 7 ⁶⁰¹³ / ₁₁₅₁₃	
			<hr/>		
			79284		
			<hr/>		
			69078		
			<hr/>		
			10206		
			<hr/>		
			5 ¹ / ₂		
			<hr/>		
			51030		
			<hr/>		
			5103		
			<hr/>		
			56133		
			<hr/>		
			46052		
			<hr/>		
			10081		
			<hr/>		
			3		
			<hr/>		
			30243		
			<hr/>		
			23026		
			<hr/>		
			7217		
			<hr/>		
			12		
			<hr/>		
			86604		
			<hr/>		
			80591		
			<hr/>		
			6013		
			<hr/>		

We find the quotient 56, which is the number of perches, with the remainder 10206: this remainder divided by the divisor would be the fraction of a perch necessary to complete the quotient; but, instead of expressing the fraction of a perch, we may express the remainder of the quotient in yards, feet, inches, and fractions of an inch. Since there are $5\frac{1}{2}$ yards in a

perch, we multiply the remainder by $5\frac{1}{2}$, and then divide the product by the divisor; we obtain the quotient, four yards, with the remainder 10081. We then multiply this by 3, and divide the product by the divisor, by which we obtain the quotient 2 for the number of feet. Multiplying the remainder by 12, and dividing by the quotient, we obtain the number of inches, which is 7: the final remainder, divided by the divisor, gives the fraction of an inch necessary to complete the quotient.

We have here reduced both the divisor and dividend to the units of the lowest class which they contain. This is not necessary: we might express them in units of any class, provided that they are both expressed in units of the *same class*. Thus, we might reduce them both to pounds and decimals of a pound, and then divide according to the rules for the division of decimals by the method of reduction explained in (335.).

$$\begin{array}{rcl} \text{£} & \text{s.} & \text{d.} \\ 47 & 19 & 5 = \text{£} 47.970 \end{array} \qquad \begin{array}{rcl} \text{£} & \text{s.} & \text{d.} \\ 2728 & 17 & 10 = \text{£} 2728.891. \end{array}$$

The object, then, is, to divide the latter decimal by the former; but, as they both contain the same number of decimal places, the decimal point may be omitted, and the numbers may be treated as whole numbers (291.). The process of division would then be as follows:—

$$\begin{array}{r} 47970 \overline{) 2728891} \quad (56.887 \\ \underline{239850} \\ 330391 \\ \underline{287820} \\ 425710 \\ \underline{383760} \\ 419500 \\ \underline{382760} \\ 357400 \\ \underline{335790} \\ 21610 \\ \underline{ 21610} \\ \text{T } 4 \end{array}$$

We have here continued the division after the remainder has been found as far as the third place of decimals. Since a unit in each place expresses the thousandth of a perch, which is less than the fifth of an inch, it is unnecessary to continue the division farther, since the purposes of such computation do not require smaller fractions of an inch. The integral places here express perches, and the decimal places fractions of a perch. If it be required to convert these fractions of a perch into yards, feet, &c., it may be done as follows:—Since there are $5\frac{1}{2}$ yards in a perch, multiply $\cdot887$ by $5\cdot5$, and we shall find the yards and fractions of a yard:—

$$\begin{array}{r} \cdot885 \\ 5\cdot5 \\ \hline 4425 \\ 4425 \\ \hline 4\cdot8675 \end{array}$$

The integer 4 here expresses the yards, and the decimal places fractions of a yard. Since it is not required, however, to attain accuracy beyond the hundredth part of a yard, we shall neglect the decimal places after the second, and shall reduce the decimals of a yard to feet by multiplying by 3: we have then

$$\cdot86 \times 3 = 2\cdot58.$$

The integer 2 here expresses the feet, and the decimal $\cdot58$ the fraction of a foot: this may be reduced to inches by multiplying it by 12; we have, then,

$$\cdot58 \times 12 = 6\cdot96.$$

The total quotient is then 56 perches, 4 yards, 2 feet, 6·96 inches. The number of inches is not equivalent to the number found by the preceding method, because of the omission of the decimal places beyond the third, in the fractions of a pound; but in practice this small quantity is altogether unimportant.

Example. We pay 1*l.* for 30 yards, 4 feet, 7 inches of a certain stuff; what sum must be paid for 658 yards, 5 feet, 11 inches, 8 lines?

It is evident that, if we could find how often 30 yards, 4 feet, 7 inches are contained in 658 yards, 5 feet, 11 inches, 8 lines, we should then know the number of pounds and fractions of a pound necessary to be paid. We have then to divide the latter length by the former, and this may be done by reducing both to units of the same class, and dividing them as whole numbers. In this case, we shall reduce both lengths to lines:—

yds. ft. ins. lines.	yds. ft. inches.	
658 5 11 8	30 4 7	13620) 285116 (20.933 = 20 <i>l.</i> 18 <i>s.</i> 8 <i>d.</i>
<u>1979</u> feet	<u>94</u> feet	<u>127160</u>
12	12	122580
<u>2359</u> inches	<u>1135</u> inches	<u>45800</u>
12	12	40860
<u>285116</u> lines	<u>13620</u> lines	<u>49400</u>
		40860

Having reduced the two lengths to lines, we divide the greater by the less, and continue the division after the remainder has been found, by annexing ciphers until we obtain three decimal places: the quotient found is 20.933, which, by the method explained in (337.), we find to be equivalent to 20*l.* 18*s.* 8*d.*

(362.) **FOURTH CASE.** When the divisor and dividend are complex numbers of different kinds.

Example. 258 lbs. 15 oz. 10 drams, of a certain alloy are bought for 325*l.* 17*s.* 10*d.*; what is the price of 1 pound? To solve this question, it is necessary to divide the given sum of money into the same number of equal parts and fractions of a part as there are pounds and fractions of a pound in the proposed weight: we must, therefore, first reduce the proposed weight to a fraction of a pound, and then divide the given sum by that fraction.

To reduce a complex number to an equivalent fraction of its principal unit, it is only necessary first to reduce it to an equivalent number of units of the lowest denomination which it contains, and then to divide it by the number of those units contained in the principal unit. In the present case, we shall first reduce the proposed weight to drams, and then divide the number of drams so found by the number of drams in a pound: the quotient will express, in the form of a mixed number, the number by which it is necessary to divide the given sum of money, and the division may be performed by the methods explained in the second case of this chapter: the process will be as follows:—

lbs.	oz.	drams.
258	15	10
16		
<hr/>		
1548		
258		
15		
<hr/>		
4143		
16		
<hr/>		
24858		
4143		
10		
<hr/>		
66298		
<hr/>		

Since there are 16 ounces in a pound, and 16 drams in an ounce, we shall find the number of drams in a pound by multiplying 16 by 16; the product is 256. To reduce the number of drams in the proposed weight to pounds and fractions of a pound, we have then only to divide it by 256: hence we find 258 lbs. 15 oz. 10 lbs. = $\frac{66298}{256}$ lbs. This fraction may be reduced to lower terms, by dividing both numerator and denominator by 2, after which it becomes $\frac{33149}{128}$: we must then divide the proposed sum of money by this fraction, which is done by multiplying it by 128, and dividing the product by 33149.

	£	s	d.
	3259	17	10
	128		
	26072		
	6518		
	3259		
s.	d.		
10	0	-	-
5	0	-	-
2	0	-	-
	6	-	-
	3	-	-
	1	-	-
	417266	2	8

	£	s.	d.
66298)	417266	2	8 (6 5 10 ³³⁸⁹² ₆₆₂₉₈
	397788		
	19478		
	20		
	389562		
	331490		
	58072		
	12		
	696872		
	66298		
	33892		

The fraction being less than a farthing, may be neglected, and the answer to the question is *6l. 5s. 10d.*

(363.) When the several examples which we have given in this chapter are considered, it will be perceived that in some cases the species of complex number, which the quotient must be, will be apparent from the very nature of the division ; but in other cases it can only be known from the conditions of the question proposed.

If the divisor be a simple or abstract number, then the quotient must be a complex number similar to the dividend ; for the effect of the operation, in that case, would be, to find such a number as, when repeated as often as there are units and parts of a unit in the divisor, would make up the dividend. Since, then, the dividend

would be made up by the repetition of the quotient, it is evident that the quotient must be a complex number of the same kind as the dividend.

If the divisor be a complex number of the same kind as the dividend, then the effect of the division is to ascertain how often the divisor should be repeated in order to make up the dividend. So far as the mere operation of division is concerned, the quotient would, in this case, be merely an abstract or simple number, and the nature of its unit can, therefore, be only discovered by the peculiar nature of the question from which it arises. In the first example given in the third case, we divided $2728\text{l. } 17\text{s. } 10\text{d.}$ by $47\text{l. } 19\text{s. } 5\text{d.}$: we should obtain by such division the quotient $56\frac{10206}{11503}$; but it would, so far as the mere process of division is concerned, be impossible to say what is the nature of the units of this quotient: in fact, the quotient would only express the number of times that the divisor must be repeated to make up the dividend. But on examining the question which gave rise to this division, we find that the divisor is the price of cutting one perch of a canal, and that the number sought is the price of cutting as many perches of the same canal as there are units in the number which expresses how often the divisor is contained in the dividend: that number is the quotient, and, therefore, the principal units of the quotient express perches. Had the question from which this division arose stated that the divisor was the price of one hundred-weight of a certain commodity, then the principal units of the quotient would be hundred-weights; but in other respects the question would remain unaltered. It is the principal units only, however, which are independent of the nature of the question; for when we convert the fractional part of the principal unit into inferior units, the method of proceeding will entirely depend upon the nature of the principal units.

If the divisor and dividend be complex numbers of different kinds, then the object of the division is to

find such a number as shall be contained in the dividend, the same number of times and parts of a time that the principal unit of the divisor is contained in it: hence it follows that in this case the quotient must be a complex number of the same kind as the dividend. We may therefore infer, generally, that the dividend and quotient are always complex numbers of the same kind, except in the case in which the divisor and dividend are complex numbers of the same kind.

BOOK IV.

PROPORTION, AND ITS PRACTICAL APPLICATIONS.

CHAPTER I.

PROPORTION.

(364.) **THERE** is no mathematical term in more familiar use, and about the meaning of which less doubt exists in its common acceptation, than the word **PROPORTION**, and yet there is no term in its scientific use respecting the exact definition of which more numerous and perplexing disputes have been maintained. These disputes, however, rest more upon the geometrical than the arithmetical use of the term. In its latter application, we shall not encounter much difficulty in fixing its precise meaning.

When we say that all the parts of a building, or of a piece of furniture, or any other structure, are in *proportion*, it is evidently meant that when its different parts, or its measurements in different ways, are compared together, no one will be found too great or too small for another. We say the human figure in a dwarf is out of proportion, because his breadth or thickness is too great for his height ; or, what amounts to the same, his height is too small for his breadth or thickness. All this implies that there exists a certain known relation, with respect to height, breadth, and thickness in the human figure, of such a nature that, when any one of these dimensions is increased, the other is similarly increased.

The common acceptation of the word **SCALE** will illustrate the notion of Proportion. A map exhibits a copy of the outline of a country, but on a reduced

scale. What is the meaning of this? If every mile in the length of the country correspond to a tenth of an inch in the length of the picture of that country exhibited on the map, it is clearly intended that every mile in the breadth shall be also expressed by the tenth of an inch on the map; or, in other words, if in the length of the country on the map there be as many tenths of an inch as there are miles in the actual length of the country, then there must be also in the breadth of the country on the map as many tenths of an inch as there are miles in its actual breadth; and, in general, in whatever direction the country be measured, there must be as many tenths of an inch measured in the same direction, on the map, as there are miles in the country itself. In such a case, the map is said to exhibit a representation of the country, but on "a reduced scale:" every part of the map is said to be "in the same proportion" as the corresponding parts of the country.

If two similar buildings be erected, having their parts in the same proportion, but one having double the height of the other, then it must follow that the breadth and depth of the one will also be double the breadth and depth of the other; that every room in the one shall have double the height, double the breadth, and double the depth of the corresponding room in the other, and so on.

(365.) From these considerations it appears that objects are said to be in the same proportion when the dimensions of the smaller are all the *same fractions* of the dimensions of the greater. Let us apply this to abstract numbers.

When two numbers, such as 7 and 8, are proposed, there is a certain relation between them, which is called their **RATIO**: this relation is expressed by the fractional part of the second number, considered as unity, which is equal to the first. In the present case, if we regard 8 as the unit, then 7 eighths of it will be equal to the first number, and accordingly the **RATIO** of 7 to 8 is

expressed by the fraction $\frac{7}{8}$. In the same manner the ratio of 8 to 7 would be expressed by the fraction $\frac{8}{7}$.

(366.) The word "ratio," therefore, always implies the comparison of two numbers in a certain determinate order: when the order is reversed the ratio is said to be inverted, and is sometimes called "the inverse ratio." Thus the ratio of 8 to 7 is the "inverse ratio" of 7 to 8.

(367.) Since the ratio of two numbers is always estimated by the fraction formed by taking the first of the two numbers as the numerator, and the second as the denominator, it follows that two ratios are equal when the two fractions thus formed are equal. Thus, the ratio of 7 to 8, and that of 14 to 16, are equal, because the fractions $\frac{7}{8}$ and $\frac{14}{16}$ are equal.

(368.) When two ratios are equal, the two pairs of numbers are said to be in *proportion*: proportion is then the equality of ratios. Thus we say the four numbers 7, 8, 14, and 16 are in proportion, and this fact is usually expressed by saying that "7 is to 8 as 14 is to 16:" the arithmetical notation by which this is expressed is as follows, $7 : 8 :: 14 : 16$.

The sign : between two numbers, therefore, expresses the ratio; and the sign :: expresses the equality of ratios.

The ordinary sign for equality is sometimes used for ratios in the same manner as for numbers: thus the above proportion would be expressed,

$$7 : 8 = 14 : 16 ;$$

and this method is perhaps to be preferred.

(369.) A ratio is also frequently expressed by the notation for fractions, since the fraction formed by its terms is the number by which the ratio is estimated: thus $7 : 8$ would be expressed $\frac{7}{8}$, and the above proportion would then be expressed thus, $\frac{7}{8} = \frac{14}{16}$. It appears, therefore, that "ratio" is only another name for a fraction, and "proportion" expresses the equality of two fractions.

Since the value of a fraction is not changed by mul-

tiplying or dividing both its terms by the same number, (210.) (211.), we may also multiply or divide both terms of a ratio by the same number without changing the ratio : thus, the following ratios are equal : —

$$\begin{array}{l} 5 : 6 \\ 10 : 12 \\ 15 : 18 \\ 20 : 24 \\ 25 : 30 \\ \&c. \&c. \end{array}$$

(370.) The first term of a ratio is called its *antecedent*, and the second its *consequent*. The antecedent, therefore, corresponds to the numerator of the fraction, and the consequent to its denominator.

(371.) Since a fraction is increased in proportion as its numerator is increased, or its denominator diminished (210.), a ratio is increased according as its antecedent is increased, or its consequent diminished.

Since two equal fractions are equally increased by multiplying their numerators by the same numbers, or by dividing their denominators by the same number (214.) (215.), two equal ratios are equally increased by multiplying their antecedents, or dividing their consequents by the same number.

*(372.) Hence two equal ratios will continue to be equal when their antecedents are multiplied by the same number, or their consequents divided by the same number, or if the antecedent of one be multiplied by a certain number, and the consequent of the other divided by the same number. For like reasons we may infer that two ratios will continue equal when the antecedents are divided by the same number, or the consequents multiplied by the same number, or the antecedent of one divided by a certain number, and the consequent of the other multiplied by the same number.

(373.) A proportion which expresses the equality of two ratios will therefore not be disturbed, but will continue to be a true proportion, though it may have been submitted to any of the above changes : the following examples will illustrate this :—

			8 : 10 = 16 : 20
Multiply antecedents by 2	-	-	16 : 10 = 32 : 20
Divide antecedents by 2	-	-	4 : 10 = 8 : 20
Multiply consequents by 2	-	-	8 : 20 = 16 : 40
Divide consequents by 2	-	-	8 : 5 = 16 : 10
Multiply one antecedent by 2, and divide the other consequent by 2	-	-	8 : 5 = 32 : 20
Divide one antecedent by 2, and multiply the other consequent by 2	-	-	4 : 10 = 16 : 40

(374.) It is frequently necessary to determine whether four proposed numbers are in proportion or not; or, in other words, whether the ratio of the first to the second be equal to the ratio of the third to the fourth. To determine this, it is only necessary to enquire whether the fractions which are equivalent to these ratios be equal. This will be known by reducing the two fractions to the same denominator, when, if they be equal, their numerators will be equal, but otherwise not. Let the four numbers proposed be 23, 27, 54, and 65. If these numbers be in proportion, then the fractions $\frac{23}{27}$, $\frac{54}{65}$, must be equal. These fractions will be reduced to the same denominator by multiplying both terms of the first by the denominator of the second, and both terms of the second by the denominator of the first (232.): the fractions will then become —

$$\frac{23 \times 65}{27 \times 65} \text{ and } \frac{54 \times 27}{27 \times 65}.$$

Now, since these have the same denominator, if they be equal, their numerators must be equal, and we shall ascertain that by actually multiplying the numbers which are here only connected by the sign of multiplication. But before we proceed to this, we shall observe that the numerators of these fractions are the products of the first and fourth, and of the second and third terms, in the proposed series of four numbers, and hence we may derive the following important

RULE.

(375.) *Four numbers will be in proportion if the*

product of the first and fourth be equal to the product of the second and third ; but, if these products be unequal, the numbers will not be in proportion.

(In the series of four numbers, the first and fourth are called *extremes*, and the second and third *means*.)

Performing the actual multiplications above indicated, we find that the product of the extremes is 1495, and the product of the means is 1458: the four numbers proposed are therefore not in proportion, the first ratio being greater than the second. Let it be required to determine whether the series 23, 27, 184, 216 are in proportion: multiplying the extremes and means we find

$$23 \times 216 = 4968$$

$$27 \times 184 = 4968.$$

The products of the extremes and means are therefore equal, and the four numbers are in proportion, so that

$$23 : 27 = 184 : 216.$$

(376.) Since the product of the means in a proportion is equal to the product of the extremes, it follows that the product of the means, divided by one extreme, will give a quotient, which is the other extreme. Thus the fourth term of a proportion may always be found by dividing the product of the second and third by the first. For the same reason, if the product of the second and third be divided by the fourth, the quotient must be the first.

In the proportion $23 : 27 = 184 : 216$, the product of the second and third is 4968 ; this, divided by 23, will give the quotient 216, which is the fourth ; also, if divided by 216, it will give the quotient 23, which is the first.

(377.) For similar reasons, if the product of the first and fourth terms of a proportion be divided by the second, the quotient will be the third ; and if it be divided by the third, the quotient will be the second.

In the proportion $23 : 27 = 184 : 216$ the product of the first and fourth is 4968 : if this be divided by

27, the quotient will be 184; and if it be divided by 184, the quotient will be 27.

(378.) From these observations, it follows that if any three terms of a proportion be known, the fourth term may always be found, for of the three known terms two must either be means or extremes. In the proportion, let these two be multiplied together, and their product found: this product divided by the other known term will always give a quotient, which is the remaining term.

If the first, second, and third terms of a proportion be 23, 27, and 184, the fourth term will be found, by multiplying 184 by 27, and dividing the product by 23; the quotient will be 216, as we have already seen.

In the same manner, if the first, second, and fourth terms be 23, 27, and 216, the third will be found by multiplying 23 by 216, and dividing the product by 27: the quotient will be 184, as already found.

(379.) Since four numbers, which are in proportion, will continue to be in proportion so long as the product of the means is equal to the product of the extremes, it follows that we may make any of the following changes in the *order* of the four terms, without destroying their proportion.

1. The means and extremes may interchange places: thus, if $23 : 27 = 184 : 216$, then we shall also have $27 : 23 = 216 : 184$, or $27 : 216 = 23 : 184$, or $184 : 23 = 216 : 27$, or $184 : 216 = 23 : 27$.

2. The places of the extremes may be transposed: thus, if we have $23 : 27 = 184 : 216$, we may infer $216 : 27 = 184 : 23$.

3. The means may be transposed: thus, if we have $23 : 27 = 184 : 216$, we may infer $23 : 184 = 27 : 216$.

4. Both means and extremes may be transposed: thus, if we have the same proportion as above, we may infer $216 : 184 = 27 : 23$.

(380.) Any change may be made in the means or extremes, which does not alter the value of their product, and such change will not destroy the proportion: thus, one mean or extreme may be doubled, and the

other halved, or, in general, one may be multiplied by any number if the other be divided by the same number. Also, if one mean be doubled, and one extreme also doubled, the proportion will be preserved, because in that case the product of the means and extremes will be both doubled; and in general, if one mean and one extreme be multiplied or divided by the same number, the proportion will be preserved, because in that case the products of the means and extremes must continue to be equal.

These observations will be found useful in abridging the computation of many arithmetical questions, involving the consideration of proportions.

(381.) If two or more proportions be given, another proportion may be obtained by multiplying the corresponding terms of the given proportions, — that is, by multiplying all the first terms together for a first term, all the second terms for a second term, and so on. Thus if the following proportions be given : —

$$3 : 5 = 18 : 30$$

$$2 : 1 = 4 : 2$$

$$4 : 3 = 8 : 6$$

we shall have the following proportion : —

$$3 \times 2 \times 4 : 5 \times 1 \times 3 = 18 \times 4 \times 8 : 30 \times 2 \times 6.$$

That this last is a true proportion, may be ascertained by actually performing the multiplications indicated, and then obtaining the products of the extremes and means, which will be found to be equal; but we may infer the truth of the general principle, of which this is merely an example, by expressing the proportions as fractions in the following manner : —

$$\begin{array}{l} \frac{3}{2} = \frac{18}{4} \\ \frac{4}{3} = \frac{8}{6} \end{array} .$$

Since the three fractions in the first column are respectively equal to the three fractions in the second column the continued product of the former must be equal to the

continued product of the latter; but the continued product of the former is,

$$\frac{3 \times 2 \times 4}{5 \times 1 \times 3}$$

and that of the latter,

$$\frac{18 \times 4 \times 8}{30 \times 2 \times 6}$$

These fractions being equal, the ratio of the numerator to the denominator of the first, must be the same as the ratio of the numerator to the denominator of the second: hence we infer :—

$$3 \times 2 \times 4 : 5 \times 1 \times 3 = 18 \times 4 \times 8 : 30 \times 2 \times 6.$$

CHAP. II.

THE RULE OF THREE.

(382.) THE method explained in the last chapter, by which, when three terms of a proportion are given, the fourth term may be found, is commonly called, in treatises on arithmetic, *THE RULE OF THREE*; and, from its extensive usefulness in the solution of arithmetical questions, was formerly called the *GOLDEN RULE*. This rule, as well as the whole doctrine and application of proportion, is indeed nothing more than a peculiar manner of expressing fractional relations, a fact which will be still more clearly illustrated in the questions, the solutions of which we shall now proceed to explain.

The rules and methods of computation which have been explained hitherto in this treatise, are sufficient for the solution of all questions whatever, which can be proposed in arithmetic; but these rules always suppose that the various arithmetical operations to be performed on the given numbers, are distinctly stated, and that the computist is only required to execute them so as to obtain correct results; but in the practical application of arithmetic, there is another difficulty which constantly presents itself, and which cannot easily be met by any very general rules. When a question is proposed, it is frequently a matter of some difficulty, especially to an unpractised computist, to collect from the conditions given, what the arithmetical operations are, which must be performed on the numbers proposed in the question, in order to obtain its solution. To discover from these conditions the series of arithmetical operations necessary to the solution, is called the *analysis* or *resolution* of the problem, and a facility in effecting such resolution can only be obtained by extensive practice in arithmetical questions, aided by such general comments on particular

questions as their circumstances and conditions may enable a judicious teacher to make.

The different circumstances and conditions which attend the resolution of arithmetical questions involving the consideration of proportion, have led to a classification of such questions under the titles of "THE DIRECT RULE OF THREE, THE INVERSE RULE OF THREE, THE COMPOUND RULE OF THREE," &c. &c.

THE DIRECT RULE OF THREE.

(383.) *Example 1. — If 25 bales of goods cost 650*l.*, what will be the price of 384 bales of the same goods?*

ANALYSIS. — In this question the price is supposed to increase or decrease in the same proportion as the number of bales increases or decreases. Now there are two numbers of bales proposed, the price of one number being given, while the price of the other number is sought. It is clear, that whatever the sought price be, the given price must have the same proportion to it as the number of bales to be obtained for the given price has to the number of bales obtained for the sought price; that is, 25 bales will be to 384 bales in the same proportion as 650*l.* is to the price of 384 bales. Let us suppose that this unknown price of 384 bales is expressed by the letter x : a proportion must exist between 25, 384, 650, and x .

COMPUTATION. — We have then —

$$25 : 384 = 650 : x.$$

We find x the fourth term of the proportion, by multiplying together the second and third, —

$$384 \times 650 = 249600$$

and dividing the product by the first (378.). The price, therefore, of 384 bales is —

$$\begin{array}{r} 25 \overline{) 249600} \\ \underline{19984} \end{array}$$

If the student should find any difficulty in compre-

hending the reasoning on which this process rests, it will be made still more evident by proceeding in the following manner:—First, find the price of a single bale. This is easily done: the price of 25 bales is 650*l.*, and the price of a single bale must therefore be the twenty-fifth part of this. If we divide 650 by 25, we find the quotient 26: the price of a single bale is, therefore, 26*l.* To find the price of 384 bales, we must multiply the price of a single bale by 384. The product of 26 and 384 is 9984: the price of 384 bales is, therefore, 9984*l.*

It will easily appear that these two processes are, in fact, identical. In the first we multiply the third term of the proportion by the second, and divide the product by the first. According to the second method, we divide the third by the first, and multiply the quotient by the second. The same operations are, therefore, performed in each case, but are performed in a different order.

The computation may also be simplified by recollecting that we may divide the first and third terms in a proportion by the same number without destroying the proportion (373.). In the present case it is obvious, upon inspection, that the first and third are both divisible by 25. If this division be effected, the proportion becomes —

$$1 : 384 = 26 : x.$$

The fourth term, x , is then found, by merely multiplying the second and third.

(384.) *Example 2.* — 87 yards, 2 feet, 4 inches of a certain canal cost 743*l.* 15*s.* 8*d.*, what will be the cost of 155 yards, 0 feet, 8 inches of the same canal?

ANALYSIS. — This question is evidently one of the same nature as the last, the numbers proposed being, however, complex numbers. There are three numbers given in order to find a fourth: two of the given numbers express certain quantities of work, the third expresses the price of one of these quantities, and the

number which is sought is the price of the other. It is evident from the conditions of the question, that the prices of the two quantities of work will be in the same proportion as the quantities themselves. The quantity of work whose price is given is to the quantity of work whose price is sought, in the same proportion as the given price is to the sought price.

COMPUTATION. — We have then the following proportion, the fourth or sought number being expressed by x : —

$$\begin{array}{rccccccc} \text{yds.} & \text{ft.} & \text{in.} & \text{yds.} & \text{ft.} & \text{in.} & \text{£} & \text{s.} & \text{d.} \\ 87 & 2 & 4 : & 155 & 0 & 8 = & 743 & 15 & 8 : x. \end{array}$$

We must here proceed to find x by multiplying the second by the third, and dividing by the first, according to the rules established for these operations on complex numbers, in Book III. chap. iii. iv. But we may abridge the process if, in the first instance, we reduce the first and second terms to simple numbers, by converting the yards and feet into inches. Proceeding according to the methods explained in Book III. chap. i. we find that

$$\begin{array}{rcccc} \text{yds.} & \text{ft.} & \text{in.} & \text{in.} \\ 87 & 2 & 4 = & 3160 \\ 155 & 0 & 8 = & 5588. \end{array}$$

Putting these numbers of inches as the first and second terms of the above proportions, it becomes —

$$\begin{array}{rcccc} & \text{£} & \text{s.} & \text{d.} \\ 3160 : 5588 = & 743 & 15 & 8 : x. \end{array}$$

The computation of the fourth term may still further be simplified by observing, that the first and second terms may be divided by 4 (380.), by which the proportion becomes —

$$\begin{array}{rcccc} & \text{£} & \text{s.} & \text{d.} \\ 790 : 1397 = & 743 & 15 & 8 : x. \end{array}$$

Multiplying 743*l.* 15*s.* 8*d.* by 1397 (346.), we find the product 1039065*l.* 6*s.* 4*d.*

Dividing this product by 790, we obtain the quo-

tient 1315*l.* 5*s.* 5 $\frac{3}{4}$ $\frac{2}{3}$ $\frac{6}{5}$ *d.*, which is, therefore, the price of 155 yards, 0 feet, 8 inches.

(385.) *Example 3.* — *If 30 quarters of corn are bought for 75*l.* 10*s.*, what will be the price of 180 quarters?*

ANALYSIS. — In this case two quantities of corn are given, and the price of one of those quantities is also given, while the price of the other is required. It is evident that the prices are in the same proportion as the quantities ; and, consequently, we can state that the first quantity is to the second quantity as the price of the first is to the price of the second, which is the number sought.

COMPUTATION. — Let x express the price of 180 quarters : we have —

$$\begin{array}{r} \text{£} \quad \text{s.} \\ 30 : 180 = 75 \quad 10 : x. \end{array}$$

In this proportion the second term is exactly divisible by the first. The process of calculation will be abridged by first dividing the second by the first, and then multiplying the quotient by the third. The quotient of the second by the first is 6. We shall, therefore, find the price sought by multiplying 75*l.* 10*s.* by 6. The product is 453*l.*

(386.) We may observe, generally, that when the first, second, and third terms of a proportion are given to find the fourth, although the operations by which it is found must always be the same, yet they may be performed in different orders, and the process may sometimes be rendered more expeditious by a proper selection of the order in which these operations are performed.

First, we may multiply the second and third together, and divide by the first.

Second, we may divide the second by the first, and multiply the quotient by the third.

Third, we may divide the third by the first, and multiply the quotient by the second.

It is evident that the same operations are performed in each case, but are taken in a different order. If it happen that either the second or the third is exactly divisible by the first, then the second or third of the above methods is preferable to the first.

In every case we should observe whether there is any number which will divide both the first and second, or the first and third, in which case such division may be made without disturbing the proportion, and its effect will be to make the calculation depend on smaller numbers.

(387.) *Example 4.* — *If the pendulum of a clock vibrate 180 times in 3 minutes, how often will it vibrate in an hour and a quarter?*

ANALYSIS. — In this case there are two portions of time given, 3 minutes and 1 hour and 15 minutes; and we are also given the number of swings of the pendulum in the first time, to find the number of swings in the second. It is evident that the number of swings or vibrations will be greater or less in proportion as the time is greater or less; and, consequently, 3 minutes is to 1 hour and 15 minutes as the number of swings in 3 minutes, *i. e.* 180, is to the number of swings in 1 hour and 15 minutes.

COMPUTATION. — In 1 hour and 15 minutes there are 75 minutes. We have, therefore, the following proportion: —

$$3 : 75 = 180 : x.$$

In this case the third term is divisible by the first. We accordingly find the quotient, which is 60, and multiply it by 75 (386.), by which we find the fourth term to be 4500.

(388.) *Example 5.* — *If 56 men are able to make 720 feet of a road in a week, how many feet of the same road would 24 men make in the same time?*

ANALYSIS. — We are here given two troops of workmen, one consisting of 56 labourers, and the other of 24. We are also given the length of road which the first troop can make in a week, and we are required to find

the length of road which the other troop can make in a week. It is evident that the two lengths of road would be greater or less in proportion as the number of labourers in each troop is greater or less. A troop of double or half the number, would perform double or half the work, and so on. Hence, if we express by x the number of feet of road which 24 men would construct in a week, we shall have the following proportion : —

COMPUTATION. —

$$56 : 24 = 720 : x.$$

In this proportion the first and second terms may be both divided by 8, by which the proportion will be reduced to —

$$7 : 3 = 720 : x.$$

To find the fourth term, we multiply the third by the second, and divide the product by the first, by which we obtain $308\frac{4}{7}$ feet, which is the length of road required.

(389.) If the preceding examples be considered, and attentively compared one with another, they will be found to agree in certain general features. In each of them there are four quantities, or numbers, contemplated, two of which are of a certain kind, and two others of another kind, each of the latter being related to each of the former in the same manner. Thus, in some of the examples, there are two quantities of work, which are the two quantities of the first kind, and the two prices of these quantities of work, which are the two quantities of the second kind. It is evident that the first price has the same relation to the first quantity of work as the second price has to the second quantity of work. In the same manner, in another example, two troops of workmen are the quantities of the first kind, and two lengths of a road which they can construct in a week are the two quantities of the second kind. In this case, also, the first length of road has the same relation to the first troop of men as the second length of road has to

the second troop of men. Again, in another example, the two quantities of the first kind are two parcels of bales of goods, and the two quantities of the second kind are the two prices of these parcels. The first of these prices has the same relation to the first parcel as the second price has to the second parcel.

The same observation will be applicable to every example which can be proposed in this class. The two terms of the first kind are given, and the first term of the second kind is also given, while the second term is sought. In every case, also, the ratio of the first term of the first kind to the second, is the same as the ratio of the first term of the second kind to the second or sought number. The two terms of the second kind may be called the *correspondents* of those of the first kind, to which they are related by the conditions of the question. Thus, when two quantities of goods are given, and the price of one of them, to find the price of the other, the two quantities of goods are the two quantities of the first kind, and the prices are their correspondents; and the two quantities of the first kind, are evidently proportional to their correspondents. Again, if two quantities of work are given, and the number of men necessary to execute the first in a day, to find the number necessary to execute the second in a day, the two terms of the first kind are the two numbers of men, and their correspondents are the quantities of work they can respectively finish in a day. In this case, also, the terms of the first kind are in the same proportion as their correspondents.

(390.) In general, then, in the class of questions to which the above examples belong we may observe,—

First, That the two terms of the first kind are in the same ratio as their correspondents; the first term being to the second term as the correspondent of the first term is to the correspondent of the second term.

Second, The correspondent of the second term, which is the number sought, is found by one of the three methods explained in (386.).

(391.) The analysis of questions of this class seldom presents much difficulty. In such questions three quantities are always given, and one sought; of the three given quantities two are always of the same kind, while the third is of a different kind, and connected with one of the two former in some manner which is distinctly expressed in the conditions of the question. Thus of the three given quantities, two may be parcels of goods, while the third is the price of one of those parcels; or two may be quantities of work, while the third is the price of performing one of them; or, again, two may be quantities of work, while the third is the time which a man would take to perform one of them, and so on. The two quantities of the same kind which are given, are what we have called the two terms of the *first kind*; but that quantity is to be taken first in order, to which the third given quantity is related. Thus if two parcels of goods be given, together with the price of one of them, we must take, as the first term of the first kind, that parcel whose price is given, while its correspondent, which is its price, will be the first term of the second kind. The student may facilitate the analysis by first writing down the two given quantities of the same kind, placing first in order that which is related to the third given quantity. Expressing the sought quantity by x , he may then write under these two terms of the first kind their correspondents. Thus, in the analysis of Example I., he would first write the quantities as follows:—

Bales	-	25	-	384
Prices	£	650	-	x .

If it is apparent, from the conditions of the question, that the first term of the first kind is to the second as the correspondent of the first is to the correspondent of the second, the analysis will be completed by stating the proportion in that form. Thus the above example would be stated thus:—

$$25 : 384 = 650 : x.$$

THE INVERSE RULE OF THREE.

(392.) The analysis of the following example will lead us to consider another class of problems.

Example 6.—If 21 men take 18 days to perform a certain work, in how many days will 7 men perform the same work, working at the same rate?

ANALYSIS.—In this case, the same quantity of work which 21 men are able to execute in 18 days, is required to be done by only 7 men. It is evident that, in proportion as the number of labourers is *diminished*, the time required for the performance of the work will be *increased*; one man will take two days to perform what two men will do in one day. In the same manner, one man would take 3, 4, or 5 days to perform what 3, 4, or 5 men would do in one day. The two terms of the first kind are in this case the two numbers of men, and their correspondents are the number of days in which they would respectively perform the work. The correspondent of the first troop is 18; let us call the correspondent of the second troop, which is sought, x .

Labourers	21	-	7
Days' work	18	-	x .

From what has been just stated, it appears that the first term of the first kind is to the second as the correspondent of the second is to the correspondent of the first; or, what amounts to the same, the second term of the first kind is to the first as the correspondent of the first is to the correspondent of the second. We have, therefore, the following proportion:—

COMPUTATION.—

$$7 : 21 = 18 : x.$$

In this proportion the second term ^a is divisible by the first. Performing this division, and multiplying the quotient by the third (386.), we find the fourth term to be 54; 7 men would, therefore, take 54 days to perform the work.

Example 7.—A steam engine moving on a level railway is capable of transporting carriages at the rate of 32 miles an hour, while carriages drawn by horses on turn

pike roads cannot exceed the average rate of 9 miles an hour. If the distance from London to Liverpool, at this rate, on the turnpike roads, is performed at present in 24 hours, in what time would the same distance be performed on a level rail-road by steam power?

ANALYSIS.—The two quantities of the first kind are the rates of travelling, and their two correspondents are the times of performing the journey.

Miles per hour	-	-	-	9	-	32
Hours from London to Liverpool				24	-	x .

It is evident that, in proportion as the rate of travelling is *increased*, the time necessary to perform the journey will be *diminished*. The second term of the first kind will, therefore, be to the first as the correspondent of the first is to the correspondent of the second.

COMPUTATION : —

$$32 : 9 = 24 : x.$$

In this proportion the first and third may be divided by 8 (380.), by which the proportion becomes —

$$4 : 9 = 3 : x;$$

from whence we find the fourth term to be $6\frac{3}{4}$. The time, therefore, on a level rail-road, would be 6 hours and 3 quarters.

(393.) In considering these examples, we find that, as in the first class, we still have under contemplation two terms of the first kind and their correspondents; but in the proportion which subsists between these terms and their correspondents, there is a striking difference. In the examples of the first class, the first term of the first kind always had the same ratio to the second as its correspondent had to the correspondent of the second. The terms of the first kind in the former examples, therefore, always occupied the *same order* in the proportion as their correspondents. However, these circumstances are now reversed: the first term of the first kind is to the second, not as the correspondent of

the first to the correspondent of the second, but as the correspondent of the second to the correspondent of the first. The terms of the first kind are still proportional to their correspondents, but they are proportional to them only when taken in a *contrary order*. When the proportion subsists under these circumstances, the terms are said to be *inversely* or *reciprocally* proportional to their correspondents; and the class of questions which come under this condition is placed under the *Inverse Rule of Three*.

(394.) In general, whenever a quantity of one kind increases in the same proportion as another quantity connected with it by the conditions of the question diminishes, the question comes under this rule. Thus, the quantity of goods which can be bought for a given sum of money increases in the same proportion as the price diminishes. If cotton were 1 shilling a yard, we should only obtain half the quantity for a given sum which we would obtain at 6 pence a yard. The number of days necessary to perform a given quantity of work will diminish in proportion as the number of men employed at it is increased. In the one case we say, that the quantity of cotton to be obtained for a given sum is *inversely* as the price per yard; and, in the other case we say, that the number of days necessary for the completion of a given work, is *inversely* as the number of labourers employed at it.

If, on the other hand, one quantity increases in the same proportion as another quantity connected with it by the condition of the question increases, the one quantity is said to be *directly* proportional to the other; and all questions involving such a condition belong to the direct rule of three. Thus, the quantity of work which a given number of men performs is *directly* proportional to the time or number of days during which they are employed: the quantity of goods which can be bought at a given price is *directly* proportional to the sum of money to be expended on them, and so on.

(395.) Fractions which have the same denominator

are directly proportional to their numerators ; for, since their denominators are the same, they consist of the same aliquot parts of the unit, and their magnitudes must be greater or less, in proportion as they contain a greater or less number of these parts. But their numerators express the numbers of those parts which they respectively contain (203.) ; the fractions are, therefore, directly proportional to those numerators. The fractions $\frac{5}{12}$ and $\frac{7}{12}$ express, respectively, 5 and 7 twelfth parts of the unit, and are evidently in the proportion of 5 to 7, or directly as their numerators.

If two fractions have the same numerator, they will consist of the same number of aliquot parts of the unit, and their magnitudes will be directly proportional to the magnitude of the parts of which they consist. Thus, if the two fractions be $\frac{1}{9}$ and $\frac{1}{10}$, each will consist of 7 parts of the unit, but one of these is a ninth part, and the other is a tenth : the magnitudes of the fractions will, therefore, be in the direct proportion of a ninth to a tenth, or of $\frac{1}{9}$ to $\frac{1}{10}$. But it is evident that the magnitude of any aliquot part of the unit will be greater in proportion as the number of parts into which the unit is divided is less. Thus, if the unit be divided into 3 parts, every part will be twice as great as if it were divided into 6 parts : thus, $\frac{1}{3}$ will be double $\frac{1}{6}$, or,

$$\frac{1}{3} : \frac{1}{6} = 2 : 1,$$

or, what is the same,

$$\frac{1}{3} : \frac{1}{8} = 6 : 3.$$

In the same manner, .

$$\frac{1}{9} : \frac{1}{18} = 10 : 9 ;$$

that is, the ninth part of the unit is greater than its tenth part, in the same proportion as 10 is greater than 9. In general, then, the aliquot parts of the unit are *inversely* as the number of parts into which the unit is divided, or inversely as the denominators of the fractions by which these aliquot parts are expressed. But it has been proved above, that fractions which have the same numerator are in the direct proportion of the aliquot

parts of unity of which they are composed ; but these aliquot parts being inversely as the denominators of the fractions, it follows that the fractions themselves are inversely as their denominators. In general, therefore, *fractions which have the same numerators are inversely as their denominators.*

(396.) In stating the proportion which results from the analysis of a question in the rule of three, it is customary so to arrange the terms, that the fourth or last shall always be the number which is sought. Although this arrangement is not necessary, yet it is so usual, that it will be advantageous to point out to the student the manner in which it may in every case be conducted. If the question be one in the direct rule of three, the first term of the proportion should be that term of the first kind whose correspondent is given, and the second term should be the other term of the first kind whose correspondent is the sought quantity : the third term will then be the given correspondent of the first term. Thus, in the first example, the first and third terms are 25 bales of goods, and 650*l.*, the price of these bales ; the second term is 384 bales ; and the fourth term is its correspondent, that is, its price.

If the question belong to the inverse rule of three, the first term in the proportion must be that term of the first kind whose correspondent is sought, and the second term that whose correspondent is given ; the third term will then be the correspondent of the second term, and the fourth the correspondent of the first term, which is the sought quantity. Thus, in the *Example 6.*, the term of the first kind whose correspondent is sought is 7 men, it being required to determine the number of days in which they can perform the work ; the second term of the first kind whose correspondent, 18 days, is given, is 21 men : the order of the proportion, therefore, is —

$$7 : 21 = 18 : x.$$

(397.) In some treatises on arithmetic, the student is directed to take, as the first and second terms of the

proportion, the first term of the first kind and its correspondent. This, however, is not a correct method of proceeding, although, in many cases, the right result will be obtained. A ratio can only subsist between two quantities *of the same kind*: thus, we can conceive a ratio subsisting between two weights, since one may be double or triple another, or may have any other assigned proportion to it; but we cannot conceive any ratio subsisting between a weight and a sum of money, which will be perceived when we consider the absurdity implied in the statement, that a certain number of pounds, ounces, and grains, are double or triple a number of shillings, pence, and farthings. It is true that a ratio may subsist between the *abstract* numbers expressing money and weight, but that ratio can only be conceived when the numbers themselves are considered apart from the quantities which they express. Thus, the number 10 has a certain ratio to the number 5, but 10 shillings has no ratio to 5 ounces. In the first example given in this chapter, the proportion which results from its analysis is, that the same ratio which subsists between 25 bales and 384 bales, also subsists between 650*l.* and the price of 384 bales. Here, the first ratio is between two weights, and the second ratio between two sums of money; but we should divest the statement of all propriety and distinct meaning, if we stated that the ratio of 25 bales to 650*l.* was the same as the ratio of 384 bales to the price of 384 bales. The absurdity of such a statement will be apparent, if, instead of using the general term ratio, we state some specific ratio: thus, to say that, 25 bales being three fourths of 650*l.* sterling, 384 bales must also be three fourths of its price, is manifestly absurd.

It will be asked, then, if such an absurdity exists in such statements of questions in the rule of three, how it happens that the operations performed under these statements produce right results? The answer is obvious. When the numerical statement is made, the numbers are frequently divested of their compound character, and, in fact, become abstract numbers: this

always happens when only one species of units enter each term of the proportion. Thus, in the first example, the units of the first and second terms express bales of goods, and those of the third, pounds sterling. When the numerical statement is made, it is not necessary to consider the nature of these several units, but only to recollect that the units of the fourth term must express pounds sterling. In this case, therefore, so far as the mere process of computation is concerned, the numbers are treated as abstract numbers, and the result will be the same whatever their units are supposed to express.

Example 8.—*A vessel at sea has provisions sufficient to supply the passengers and crew with their usual daily rations for 20 days; but it cannot complete its voyage in less than 35 days. It is required to determine what diminution must be made in the daily ration of each individual, so as to make the provisions last till the voyage has been completed.*

ANALYSIS. — In this case, the same quantity of provisions which, at the ordinary rate, would last only 20 days, is required to last for 35 days. It is evident that the daily rations must, therefore, be diminished, and also that they must be diminished in exactly the same proportion as the number of days they are required to last is increased. Thus, if they were required to last double the time, each individual should only be supplied with half the quantity per day; in other words, the magnitude of the daily rations will be *inversely* as the times they are required to last: hence we have the following statement: —

Days	-	20	-	35
Rations per day		1	-	x .

Here we have used 1 to express the ordinary ration, and x to express the diminished ration which we seek; and, from what has been just explained, it is clear that x is less than 1, in the same proportion as 20 is less than 35: hence we have the following proportion: —

$$35 : 20 = 1 : x.$$

COMPUTATION.— The first two terms of this proportion are divisible by 5, by which it is reduced to—

$$7 : 4 = 1 : x.$$

The sought number is, therefore, $\frac{4}{7}$, and the daily ration will be four sevenths of the ordinary ration. This question might also be solved without the consideration of proportion. The actual quantity of provisions being 20, its supply may be expressed by the number 20; and, since this must last 35 days, the daily consumption must be its 35th part: the fraction, therefore, of the ordinary daily consumption which must be used in order to last 35 days, is $\frac{20}{35} = \frac{4}{7}$.

COMPOUND RULE OF THREE.

(398.) In the examples which we have hitherto explained, the number of given terms which occurs does not exceed 3, from which a fourth is required to be found by computation. We shall now investigate some questions in which a greater number of given numbers occurs.

Example 9.— *If 20 men require 18 days to build 500 feet of wall, in how many days can 76 men build 1265 feet of a similar wall?*

ANALYSIS.— In this Question there are 3 pairs of quantities of different kinds contemplated: there are, *first*, two troops of workmen; *second*, two pieces of work; *third*, two numbers of days. The first and second pairs of numbers are given, and one of the numbers of the third pair is given, while the other is sought. Expressing the sought number of days by x , the numbers contemplated are as follows:—

Men	-	20	-	76
Feet of wall	-	500	-	1265
Days	-	18	-	x .

It will be perceived, that the three numbers in the first column are related to each other in the same manner as the three numbers in the second column: in each of them there are expressed a number of men, the work

they can perform, and the time of performing it; but in the first column all the numbers are given, whereas in the second column one of the numbers is sought. From the first column we shall be able to derive the means of finding the sought number in the second column.

Since 500 feet of wall are built in 18 days by the first troop of builders, one day's work of this troop will be found by dividing 500 by 18. Having thus found one day's work of the first troop, we shall find the daily work of a single builder by dividing the work of the first troop by 20. The daily work of a single builder being thus found, we shall find the daily work of the second troop by multiplying the daily work of one builder by the number of builders in the second troop, which is 76. The daily work of the second troop being thus found, we shall find how often it is contained in 1265 feet by dividing the latter by it; the quotient will evidently be the number of days' work for the second troop in 1265 feet of wall, and will, therefore, be the number sought.

COMPUTATION. — I. Divide 500 by 18; the quotient will be $27\frac{1}{3}$, or $27\frac{2}{3}$ feet: this is the daily work of the first troop.

II. Divide $27\frac{2}{3}$, the daily work of the first troop, by 20; the quotient will be $\frac{2}{3}\frac{2}{3}$, or $\frac{2}{3}$ feet, which is the daily work of one man.

III. Multiply $\frac{2}{3}$ feet, the daily work of one man, by 76, the number of men in the second troop; the product is $105\frac{2}{3}$ feet, which is the daily work of the second troop.

IV. Divide 1265 feet by $105\frac{2}{3}$ feet; the quotient is $11\frac{8}{9}$, which is, therefore, the number of days' work for the second troop in 1265 feet of wall. This number is within a minute fraction of 12 days.

The preceding analysis of the question has been made without the immediate consideration of proportion. We shall now consider it under another point of view.

ANALYSIS. — In the question as announced, three ratios

are contemplated; *first*, the ratio between the number of labourers in the first and second troop; *secondly*, the ratio between the magnitudes of the two pieces of work; and, *thirdly*, the ratio between 18 days and the number of days sought. To simplify the investigation, we shall first enquire what number of days the second troop would require to complete the first piece of work. Since the number of days necessary to perform any work increases in the same proportion as the number of labourers diminishes, this will be a question in the inverse rule of three, and will resolve itself into the following proportion: — As the number of workmen in the second troop is to the number in the first troop, so is 18 days to the number of days which the second troop would require to complete 500 feet of the work.

Having found this number, we shall easily discover the number of days which the second troop would take to complete 1265 feet of work. In this case the number of days necessary to perform the work is proportional to the work to be done: hence we infer, that 500 feet is to 1265 feet as the number of days necessary to complete 500 feet to the number of days necessary to complete 1265 feet. This is obviously a question in the direct rule of three.

From this analysis it will be perceived that, in order to solve the question proposed, it is necessary to solve another question; viz., to discover the time which the second troop would take to perform the first piece of work.

COMPUTATION.—I. Let x express the number of days in which 76 men would complete 500 feet of work: we have the following proportion: —

$$76 : 20 = 18 : x.$$

The first and second terms of this proportion being divisible by 4, it is reduced to —

$$19 : 5 = 18 : x.$$

Multiplying the second and third together, and dividing by the first, we find the quotient $4\frac{1}{5}$.

II. Expressing now by x the number of days which the second troop would take to perform 1265 feet, we have the following proportion : —

$$500 : 1265 = 4\frac{1}{10} : x.$$

The first and second terms of this proportion being divisible by 5, it is reduced to —

$$100 : 253 = 4\frac{1}{10} : x.$$

Reducing the third term to an improper fraction (235.), and multiplying by the second, we find $\frac{235 \times 41}{100}$; this must be divided by 100, which gives $\frac{235 \times 41}{10000} = 11\frac{187}{1000}$, the same result as found in the preceding analysis, and which expresses the number of days and fractions of a day necessary for the second troop of workmen to execute 1265 feet of work.

(399.) *Example 10.*—In 6 days 126 acres of meadow are mowed by 14 mowers. It is required to determine how many acres can be mown in 3 days by 16 mowers.

ANALYSIS.—We have here under consideration two troops of mowers working for two distinct times, in which one of the troops mow 126 acres, and the other a number of acres which we are required to discover. The numbers involved in the question are as follows, expressing the sought number of acres by x :—

Mowers	-	14	-	16
Acres	-	126	-	x
Days	-	6	-	3.

As in the former example, we shall find from the first column the daily work of one labourer ; by dividing the number of acres by the number of mowers, we shall find the work of one mower in 6 days ; and, by dividing this work by 6, we shall find the daily work of one mower.

Having found the daily work of one mower, we shall find the daily work of the second troop, consisting of 16 mowers, by multiplying the daily work of one mower by 16 ; and we shall find their work in 3 days by multiplying their daily work by 3.

COMPUTATION.—I. Divide 126 by 14, and we find the quotient 9, which is, therefore, the number of acres mown in 6 days by one mower.

II. Divide 9 by 6, and the quotient is $\frac{3}{2} = 1\frac{1}{2}$. The daily work of each mower is, therefore, an acre and a half.

III. Multiply $1\frac{1}{2}$ by 16; the product is 24, which is therefore the daily work of 16 mowers.

IV. Multiply 24 by 3: the product is 72, which is the work of 16 mowers in 3 days, and is the answer to the question.

Example 11.—A regiment, consisting of 939 men, consume 351 quarters of wheat in 336 days; how many men will consume 1404 quarters in 112 days?

ANALYSIS.—Expressing the number of men required by x , we shall have the following statement:—

Soldiers	-	939	-	x
Quarters of corn		351	-	1404
Days	-	336	-	112.

From the first column we shall find the quarters of corn consumed by one man in 336 days, by dividing the total number of quarters consumed by the number of soldiers; we shall next find the daily consumption of one soldier by dividing the quotient last found by the number of days, 336.

Since the required number of men consume 1404 quarters in 112 days, we shall find their daily consumption by dividing 1404 by 112.

Having found the daily consumption of the sought number of men, we shall find that number by dividing it by the daily consumption of one man.

COMPUTATION.—I. Divide 351 by 939; the quotient is $\frac{351}{939}$, which, reduced to its least terms, becomes $\frac{117}{313}$, which is the fraction of a quarter consumed by one man in 336 days.

II. To find the fraction consumed by one man in one day, we must divide this by 336, which is done by multiplying its denominator by that number (215.); the result is $\frac{117}{67368}$. This, then, is the fraction of a quarter daily consumed by a single soldier.

III. Divide 1404 by 112; the quotient is $12\frac{1}{2}$,

which is, therefore, the number of quarters daily consumed by the number of men required.

IV. Divide $12\frac{15}{28}$ by $\frac{117}{105168}$; the quotient is 11268*, which is, therefore, the number of men required.

(400.) If each of the preceding examples be attentively considered, we shall perceive that, although they differ in words, and in the particular numbers involved in them, they are still identical in their essential parts, and all belong to the same class: in all of them certain *agents* are contemplated, which are employed in producing certain *effects*, in a certain *time*. In each there are *two* numbers of *agents*, *two* effects, and *two* times, considered, so that six quantities or numbers enter each question. In every case *five* of these numbers are given, and it is the object of the question to determine the *sixth*.

In the *Example* 9. the *agents* are two troops of builders; the *effects* proposed are certain numbers of feet of wall built by these troops; and the *times* are the numbers of days which the troops respectively require to produce the effects. The numbers of men in the troops are given; the work which they perform is also given, as well as the number of days in which one of them performs the work; while the number of days in which the other performs it is sought. The data here are, therefore, the two numbers of agents, the two effects, and the time of performing one of them; while the sought quantity is the time of performing the other.

In *Example* 10. two bands of mowers, the number of acres they mow respectively, and the number of days in which they perform the work are considered. The number of mowers in each band, the number of days which each band works, and the number of acres which one band mows, are severally given; while the number of acres mown by the other band is sought. In this case, therefore, the two sets of agents, the times which they take to produce the effects, and the effect produced by one of them, are given; while the effect produced by the other is sought.

* The exact quotient is 11267 $\frac{3175}{3738}$, being less than 11268 by a small fraction.

In *Example 11*. the *agents* are soldiers ; the *effects*, the consumption of certain numbers of quarters of corn ; and the *times*, the days in which these quarters are consumed. In this case the sought quantity is one of the numbers of soldiers ; the effects, times, and the other number of soldiers are given.

As this class of questions is very extensive, and embraces a great number of the examples which are usually given for exercise in the *Compound Rule of Three*, it may be worth while here to investigate a general rule for the solution of all such questions.

Let us take *Example 10*. Supplying the place of the sought quantity by 72, which it was found to be by the computation, the statement would be as follows : —

Agents	-	14	-	16
Effects	-	126	-	72
Times	-	6	-	3;

by which we mean to express that 14 agents produce the effect 126 in the time 6 ; and, also, that 16 agents produce the effect 72 in the time 3. If we divide the effect produced by the first number of agents by 14, we obtain the fraction $\frac{126}{14}$, which is the effect produced by a single agent in the time 6. If we divide this again by 6, which is done by multiplying its denominator by 6, we shall find the effect produced by 1 agent in the time 1 : this effect is, therefore, $\frac{1 \cdot 6}{14 \times 6}$.

In like manner, from the second column we find, that the work performed by 1 agent in 3 days is $\frac{72}{16}$, and, therefore, the work performed by 1 agent in 1 day is $\frac{72}{16 \times 3}$.

Since, then, the fractions $\frac{126}{14 \times 6}$ and $\frac{72}{16 \times 3}$ both express the effect produced by 1 agent in the time 1, they must be equal ; and, therefore, the numerator is to the denominator of the first as the numerator to the denominator of the second ; and we have the following proportion : —

$$126 : 14 \times 6 = 72 : 3 \times 16.$$

In this proportion the product of the means is equal

to the product of the extremes, and, therefore, we have —

$$126 \times 3 \times 16 = 72 \times 14 \times 6.$$

It appears, then, that the continued product of the *first* and *last* numbers of the *first* column, and the *middle* number of the *second*, is equal to the product of the *first* and *last* numbers of the *second* column, and the *middle* number of the *first*. The same reasoning will show that this will always be the case in questions where *agents*, *effects*, and *times* are considered.

The manner in which the numbers should be connected by multiplication may always be shown by the following

RULE.

(401.) *First, arrange the numbers expressing the agents, effects, and times in two columns, as already explained; then transpose the numbers expressing the effects, placing the first in the second column, and the second in the first. After this change, the product of the three numbers in the first column will be equal to the product of the three numbers in the second column.*

Thus, in the preceding example, the two columns will first stand thus:—

Agents	-	14	-	16
Effects	-	126	-	72
Days	-	6	-	3.

Transposing the numbers expressing the effects, the arrangement becomes —

Agents	-	14	-	16
Effects	-	72	-	126
Days	-	6	-	3.

The product of the three numbers in the first column is now equal to the product of the three numbers in the second.

(402.) We have here assumed the six quantities as all known. Now, any one of the six may be the number sought in the question, while the other five are given;

and the method by which such sought quantity may be found is perceivable at once, by considering the equality of the two products above. In the second question from which we have taken the numbers used in this illustration, the quantity sought was the effect produced by the second set of agents. We know, then, by what has been just proved, that this effect must be expressed by a number which, if multiplied by the product of the two given numbers in the first column, viz. 14 and 6, would give a product equal to the continued product of the three numbers in the second column, viz. 16, 126, and 3. The statement would be as follows:—

Agents	-	14	-	16
Effects	-	x	-	126
Days	-	6	-	3.

The products which are equal are as follows:—

$$16 \times 126 \times 3 = 14 \times 6 \times x.$$

Since x is such a number that, if multiplied by 14×6 , we should obtain a product equal to $16 \times 126 \times 3$, it follows that the latter number, divided by 14×6 , should give a quotient which will be the number x . To find x we have, therefore, only to find the continued product of 16, 126, and 3, and to divide it by 14×6 . This process is expressed thus:—

$$x = \frac{16 \times 126 \times 3}{14 \times 6}.$$

By generalising the above results, we shall find that all questions involving the consideration of agents, effects, and times,—provided they involve nothing more,—may be solved by the following

RULE.

(403.) *Write in two columns, as usual, the agents, effects, and times, transposing, however, the place of the effects, so that the effects of each set of agents shall be found in the column with the other set. Supply the place of the number required by the letter x : there will then*

be three given numbers in one column, and two in the other. Find the continued product of the former, and divide it by the product of the latter; the quotient will be the number sought.

Example 12.—If the capital, 100*l.*, invested in trade, gain 16*l.* in 12 months, what capital would gain 20*l.* in 19 months?

ANALYSIS. — In this case the capital is the *agent*, and the profit the *effect*. According to the above rule, the statement would be as follows: —

Agents	-	£ 100	-	<i>x</i>
Effects	-	£ 20	-	16
Times	-	12	-	19.

COMPUTATION: —

$$x = \frac{100 \times 20 \times 12}{16 \times 19} = \frac{24000}{304} = £78 \ 18 \ 11\frac{1}{2}.$$

Example 13. — If 60 bushels of oats are consumed by 24 horses in 40 days, in how many days will 48 horses consume 30 bushels?

ANALYSIS. — In this case the horses are the *agents*, the oats consumed the *effects*, and the number of days the *times*. According to the rule, the statement is as follows: —

Agents	-	24	-	48
Effects	-	30	-	60
Times	-	40	-	<i>x</i> .

COMPUTATION: —

$$x = \frac{24 \times 30 \times 40}{48 \times 60} = 10 \text{ days.}$$

(404.) When a question of this class has been solved, and the six numbers engaged in it all become known, the teacher can form out of them six distinct questions, including that originally proposed; for it is evident that any one of the six numbers may be sought, the other five being given. In *Example 10.*, taking successively as the sought number, the six numbers engaged in it, in the following order: — 14, 126, 6, 16, 72, 3, we should have the following six questions: —

QUESTION I. — *If 16 mowers mow 72 acres in 3 days, how many mowers will mow 126 acres in 6 days?*
Answer 14.

QUESTION II. — *If 16 mowers mow 72 acres in 3 days, how many acres will 14 mowers mow in 6 days?*
Answer 126.

QUESTION III. — *If 16 mowers mow 72 acres in 3 days, how many days will be required for 14 mowers to mow 126 acres?* Answer 6.

QUESTION IV. — *If 14 mowers mow 126 acres in 6 days, how many mowers would mow 72 acres in 3 days?*
Answer 16.

QUESTION V. — *If 14 mowers mow 126 acres in 6 days, how many acres will 16 mowers mow in 3 days?*
Answer 72.

QUESTION VI. — *If 14 mowers mow 126 acres in 6 days, how many days will be required for 16 mowers to mow 72 acres?* Answer 3.

By observing this, the teacher will have the advantage not only of multiplying numbers for the exercise of his pupils, but also of making the results of different pupils verify each other.

(405.) In questions of this kind, when the numbers proposed are complex numbers, they should be reduced to simple numbers previously to the calculation; and those of the same kind should always be reduced to the same class of units. Thus, if two numbers expressing time enter the question, they should be both expressed either in days or in hours, &c.; but one must not be expressed in days while the other is expressed in hours; and the same observation will extend to other complex numbers.

(406.) Many questions may be brought under the preceding rule, although the consideration of time may not be immediately involved in them. Sometimes distance travelled over is considered; in which cases such distances may be conceived as taking the places of the times in the rule.

Example 14. — A carrier transports 60 cwt. to the

distance of 20 miles for 14*l.* 10*s.* ; what weight should he carry 30 miles for 5*l.* 8*s.* 9*d.* ?

ANALYSIS. — In this case the money may be considered as the *effect* produced by the transport, the goods as the *agent*, and the distance which it is carried as the *time*, since it is evident that, if the rate of transport be uniform, we might substitute the time for the distance. Thus, if the transport was effected at the rate of a mile every half hour, we might substitute 20 and 30 half hours, instead of 20 and 30 miles, in the question. The statement, therefore, according to the rule, will be as follows : —

Agents	-	60		<i>x</i>
Effects	-	£14 10 0	£5 8 9	
Miles	-	20	-	30.

It will be necessary to reduce the two sums of money to the same denomination. The first sum is equal to 290 shillings, and the second to 108 shillings and 9 pence ; but 9 pence being $\frac{3}{4}$ of a shilling, it will be equal to the decimal 0.75 : therefore, 5*l.* 8*s.* 9*d.* = 108.75. Hence, transposing the effects, the statement becomes —

Agents	-	60	
Effects	-	108.75	290
Miles	-	20	30

COMPUTATION : —

$$x = \frac{20 \times 60 \times 108.75}{30 \times 290} = 15.$$

The number required is, therefore, 15 cwt.

(407.) It sometimes happens that questions of this class are rendered apparently complex, by involving in them the condition that the effects produced by the agents shall be produced at so many hours per day, making the two sets of agents work different numbers of hours per day. This complexity, however, may immediately be removed by previously finding the total number of hours which each set of agents works, and then omitting altogether the consideration of the number of days. Thus, if one set of agents works 10 days

at 6 hours a day, while the other set works 8 days at 10 hours a day, it will be sufficient to state, that the first set of agents works for 60 hours, and the second set for 80 hours. Such questions may, therefore, be reduced to the above class, provided the number of days and hours per day are given for both sets of agents.

Thus, *Example 10.* might have been announced in the following manner : —

If 14 mowers mow 126 acres in 6 days, working at the rate of 12 hours per day, how many acres would be mown by 16 mowers, working for 4 days at the rate of 9 hours a day?

The first set would, in this case, work for 72 hours, and the second set for 36 hours, and the statement would be made as follows : —

Mowers	-	14	-	16	
Acres	-	<i>x</i>	-	126	$x = \frac{16 \times 126 \times 36}{14 \times 72} = 72.$
Hours	-	72	-	36.	

If it happen, however, that in such a question the hours per day, or the number of days, be the number required, we cannot adopt this method, and the question will belong to a class in which there will be a greater number of given quantities than in the examples already explained.

Example 15. — If 14 mowers mow 126 acres in 6 days, at 12 hours per day, how many hours a day must 16 mowers work in order to mow 72 acres in 9 days?

ANALYSIS. — The statement will be as follows : —

Agents	-	14	-	16	
Effects	-	126	-	72	
Days	-	6	-	9	
Hours per day		12	-	<i>x</i> .	

We shall find, from the first column, the effect which one agent can produce in an hour. If we divide 126 by 14, we shall find the number of acres mown by one mower in 6 days at 12 hours a day. If we divide this again by 72, or by 6×12 , we shall find the quantity

mown in one hour : this will, therefore, be $\frac{126}{14 \times 6 \times 12}$. Proceeding in the same way with the second column, we shall divide 72 by 16, to find the work of one man in the whole time, and again divide this by the total number of hours ; to find the work of one man in one hour. Supposing, then, the number of hours per day during which the second band are employed, to be expressed by x , the total number of hours would be 9 times this number, and would, therefore, be expressed by $9 \times x$. The work of one man in one hour will be found by dividing $\frac{72}{16}$ by $9 \times x$; but this division is made by multiplying the denominator of the fraction by the multiplier. Hence the work of one man in one hour, as deduced from the second column, would be expressed by $\frac{72}{16 \times 9 \times x}$. Since the fractions found to express the work of one man in one hour from each column must be equal, their numerators must have the same ratio to their denominators, and hence we have the following proportion : —

$$126 : 14 \times 6 \times 12 = 72 : 16 \times 9 \times x.$$

The product of the means must be equal to the product of the extremes : therefore we have —

$$14 \times 6 \times 12 \times 72 = 126 \times 16 \times 9 \times x.$$

The required number is, therefore, such a number that if it were multiplied by $126 \times 16 \times 9$, it would give a product equal to $14 \times 6 \times 12 \times 72$; and, therefore, the latter product, divided by the former, must give a quotient equal to x . Hence we have —

$$x = \frac{14 \times 6 \times 12 \times 72}{126 \times 16 \times 9} = 4.$$

The required number of hours per day is, therefore, 4.

It appears, therefore, that to questions of this kind, the rule (403.) already explained may be extended.

RULE.

(408.) *All questions in which the effects produced*

by agents working a certain number of days and hours per day are considered, may be solved in the following manner : — Place the numbers expressing the agents, effects, days, and hours per day in two columns, transposing, however, the numbers expressing the effects, and using x to express the number sought. There will then be four given numbers in one column and three in the other. Divide the continued product of the former by the continued product of the latter, and the quotient will be the number required.

Example 16. — If a troop of 15 labourers, working 10 hours a day, take 18 days to build 450 feet of wall, it is required to determine how many workmen, working for 12 hours a day, would build 480 feet of the same wall in 8 days ?

ANALYSIS. — Expressing the required number by x , the statement, according to the rule, will be as follows : —

Agents	-	15	-	x
Effects	-	480	-	450
Days	-	18	-	8
Hours per day	-	10	-	12

COMPUTATION : —

$$x = \frac{15 \times 480 \times 18 \times 10}{450 \times 8 \times 12} = 30.$$

Having found the answer to such a question as this, the teacher can frame out of the result seven other questions, by taking successively, as the number sought, each of the numbers which are given in the above example. Thus, if he take 480 as the number sought, the question would be as follows : —

If 15 labourers, working for 10 hours a day, take 18 days to build 450 feet of wall, how many feet of wall will 30 labourers build, working for 8 days at 12 hours a day ? Answer 480.

Again, if 12 be taken as the number sought, the question would be as follows : —

If 15 labourers, working 10 hours a day for 18 days, build 450 feet of wall, how many hours a day must 30

labourers work in order to build 480 feet of wall in 8 days?

(409.) In the last two examples seven numbers are given to find an eighth. Questions, however, of still greater complexity may be proposed. Although such cases rarely happen, either in commerce or science, yet they are useful as arithmetical exercises.

Example 17. — 500 men, labouring 12 hours a day, are engaged for 57 days in cutting a canal, which measures 1800 yards in length by 7 in breadth, and 3 in depth: it is required to know how many days will be necessary for 860 men, working 10 hours a day, to cut a canal measuring 2900 yards in length, by 12 in breadth, and 5 in depth, but which is worked through a soil which is 3 times more difficult than the former one?

ANALYSIS. — This question is resolved into the following statement: —

Men	-	500	-	860
Length	-	1800	-	2900
Breadth	-	7	-	12
Depth	-	3	-	5
Difficulty	-	1	-	3
Days	-	57	-	<i>x</i>
Hours per day	12	-	10	

In this case the *effect* will depend conjointly on the length, breadth, and depth of the canal, and the difficulty of the soil. We shall find the number of cubic yards to be cut, by each set of men, by multiplying the number of yards in the length, breadth, and depth (313.). The number, therefore cut by the first troop, will be $1800 \times 7 \times 3$, and by the second troop $2900 \times 12 \times 5$. But the second troop work through a soil 3 times more difficult, which will cause them, therefore, 3 times as much labour, and the effect will be the same as if they had cut 3 times the quantity of soil of a quality like that through which the first troop worked. We shall, therefore, express the labour of the second troop by estimating it at 3

times as many cubic yards as they actually cut: their labour, therefore, will be expressed by $2900 \times 12 \times 5 \times 3$. It appears, therefore, that the *effect*, in this case, produced by each troop is found by multiplying together the second, third, fourth, and fifth numbers in their respective columns; so that the statement will become as follows : —

Men	-	500	-	860
Effect	{	Length	1800	- 2900
		Breadth	- 7	- 12
		Depth	- 3	- 5
		Difficulty	1	- 3
Days	-	-	57	- x
Hours <i>per day</i>	-	12	-	10

But by rule (408.) we must transpose the effects : the statement, therefore, becomes —

Men	-	500	-	860
Effect	{	Length	2900	- 1800
		Breadth	12	- 7
		Depth	- 5	- 3
		Difficulty	3	- 1
Days	-	-	57	- x
Hours <i>per day</i>	-	12	-	10

The continued product of all the numbers in the first column, divided by the continued product of those in the second column, will give a quotient which will be the number sought.

COMPUTATION : —

$$x = \frac{500 \times 2900 \times 12 \times 5 \times 3 \times 57 \times 12}{860 \times 1800 \times 7 \times 3 \times 1 \times 10} = 549.$$

CHAP. III.

INTEREST. — DISCOUNT. — PROFIT AND LOSS. — BROKERAGE. —
COMMISSION. — INSURANCE. — TARE AND TRET. — PARTNER-
SHIP.

(410.) THERE are several classes of arithmetical problems which are of frequent occurrence in commerce, and which fall under the denominations of INTEREST, DISCOUNT, PROFIT AND LOSS, BROKERAGE, COMMISSION, INSURANCE, TARE AND TRET, &c., all of which are only applications of the rule of three; and the principles on which they are solved, are, therefore, fully explained in the last chapter. Their occurrence in the common affairs of life is, however, so frequent, and the calculations depending on them so important and useful, that they are generally noticed as separate commercial rules.

In all these classes of problems, the object is generally to compute a certain small fractional part of a gross amount, whether of money or merchandise, to be appropriated to some specific purpose, and this part, therefore, always increases or diminishes in proportion to the gross amount of which it is a given fraction.

INTEREST.

(411.) When a sum of money is lent by one individual to another, the latter pays to the former a certain stipulated sum for the use of the money so lent; this sum is called INTEREST, and the money lent is called the PRINCIPAL.

The amount of interest is computed in proportion to the amount of principal lent, and to the time it continues in the hands of the borrower. Thus the interest on a given sum for ten years, will be ten times the interest on the same sum for one year, and the interest on 100% for one year will be ten times the interest on 10% for the same time.

Example 1.—If the interest on 357*l.* 10*s.* for three years be 53*l.* 12*s.* 6*d.*, what will be the interest on 68*l.* 5*s.* for five years?

ANALYSIS. — The agents are, here, the two principals, the effects are the interests given and the interest sought, and the times are 3 and 5 years. We have, therefore, in conformity with what has been explained in the preceding chapter, the following statement:—

Agents	-	£357 10 0	£68 5 0
Effects	-	53 12 6	- x
Years	-	3	- 5

Transposing the effects, and converting the shillings into decimals of a pound, the statement is as follows:—

Agents	-	357·5	-	68·25
Effects	-	x	-	53·625
Years	-	3	-	5

COMPUTATION.—Multiplying the three numbers in the second column, we obtain the product 18299·53125. and multiplying the first and third in the first column, we obtain the product 1072·5. Dividing the former by the latter, we obtain the quotient 17·0625, which by converting the decimals into shillings and pence, gives 17*l.* 1*s.* 3*d.*, which is the interest of 68*l.* 5*s.* for five years, on the supposition that the interest on 357*l.* 10*s.* for three years would be 53*l.* 12*s.* 6*d.*

In this example the use made of the principal 357*l.* 10*s.*, and its interest for three years, is merely to fix the *rate of interest* which should be paid for the principal 68*l.* 5*s.*; but it is usual in practice to express the *rate of interest*, not by an uneven principal or an uneven time, but always by the principal of 100*l.* invested for one year; and consequently all questions in interest involve the consideration of the interest of 100*l.* for a year, which is therefore called invariably the *rate of interest*. Thus, if for every 100*l.* principal, 5*l.* be paid for every year it is lent, we say that the rate of interest is five per cent. per annum, or simply five per cent., the time being always understood to be one year.

Example 2. — Let it be required to determine the rate

of interest when the interest on 357l. 10s. for three years amounts to 53l. 12s. 6d.

ANALYSIS. — The question here proposed is to determine what the interest on 100*l.* is for 1 year, if the interest on 357*l.* 10*s.* be 53*l.* 12*s.* 6*d.* for 3 years. The statement is as follows : —

Agents	-	£357 10 0	£100
Effects	-	53 12 6	<i>x</i>
Years	-	3	1.

Converting the sums of money into decimals of a pound, and transposing the effects, the statement becomes as follows : —

Agents	-	357·5	-	100
Effects	-	<i>x</i>	-	53·625
Years	-	3	-	1.

COMPUTATION. — Multiplying the numbers in the second column, we obtain the product 5362·5, by merely moving the decimal point two places to the right. By multiplying the first and third terms of the first column, we obtain the product 1072·5. Dividing the former by the latter, we obtain the quotient 5: the rate of interest is, therefore, 5 per cent. per annum.

Example 3. — *What is the interest of 4500*l.* for 2 years and 5 months at 7 per cent ?*

ANALYSIS. — The statement of this question is as follows : —

Agents	-	£100	-	£4500
Effects	-	7	-	<i>x</i>
Time	-	1	-	27 5 ^m .

Converting the times into simple numbers, by reducing both to months, and transposing the effects, the statement becomes —

Agents	-	£100	-	£4500
Effects	-	<i>x</i>	-	7
Months	-	12	-	29.

COMPUTATION. — The product of the numbers in the second column is 913500, and the product of the first and third in the first column is 1200. Dividing the

former by the latter, the quotient is 761·25, which, converted into pounds and shillings, is 761*l.* 5*s.*, which is therefore the interest required.

(412.) In all cases where, after the columns are arranged for computation, numbers are found in each column which are divisible by the same number, the computation may be abridged by previously dividing them by that number. In the preceding example, the first terms in each column may be divided by 100, by omitting the ciphers.

The most common class of questions in interest is that in which it is required to find the total amount of interest for a given time, at a given rate, and for a given principal. It will be, therefore, advantageous to investigate a general rule for the solution of every such question.

Let us suppose that the principal, whatever it may be, is expressed by *P*, the rate of interest by *R*, and the time expressed in years by *Y*. The analysis of the question would then lead to the following statement : —

Agents	-	100	-	<i>P</i>
Effects	-	<i>R</i>	-	<i>x</i>
Time	-	1	-	<i>Y</i> .

Transposing the effects this would become, —

Agents	-	100	-	<i>P</i>
Effects	-	<i>x</i>	-	<i>R</i>
Time	-	1	-	<i>Y</i> .

We should, accordingly, by what has been already explained, discover the amount of interest sought, which is here expressed by *x*, by obtaining the continued product of the numbers here expressed by *P*, *R*, and *Y* and dividing that product by 100 : hence the following

RULE I.

(413.) *To find the interest of any principal for any number of years at a given rate of interest, multiply the principal by the rate, and the product by the number of years : divide the product thus found by 100, and the quotient will be the interest sought.*

The principal and rate, if they are not whole numbers of pounds, should, in this case, be expressed as decimals of a pound; and if the time be not a complete number of years, it should be expressed in decimals of a year. The method of reducing sums of money to decimals of a pound has been already explained. If the time be not a round number of years, it will generally be expressed in years, months, and days. In such cases it is sometimes more convenient to express the time in months or days only than in decimals of a year; but, in that case, it would be necessary also to express in the same denomination, that is, in months or days, the time which is expressed by 1 in the first column: the statement would, therefore, be modified in the following manner for months:—

Agents	-	100	-	P
Effects	-	x	-	R
Time	-	12	-	M.

In this case M stands for the number of months for which the interest is to be calculated; hence we would obtain the following

RULE II.

(414.) *To compute the total interest on any given principal, at a given rate, for a given number of months, multiply the principal by the rate, and the product thus obtained by the number of months, and divide the number thus found by 1200: the quotient will be the amount of interest sought.*

If the time be expressed in years and months, the interest may either be calculated separately for the years and months, by Rules I. and II., or the whole time may be reduced to months, and the calculation made by Rule II. alone. If the time be expressed in days, the statement must be again modified thus:—

Agents	-	100	-	P
Effects	-	x	-	R
Time	-	365	-	D.

and we obtain the following

RULE III.

(415.) *To find the interest of any given principal, at any given rate, for any given number of days, multiply the principal by the rate, and the product by the number of days, and divide the number thus found by 36500.*

If the time be expressed in years, months, and days, the interest for the years may be calculated by Rule I., for months by Rule II., and for the days by Rule III., and the sums thus obtained added together; or the years and months may be reduced to months, and the interest for them calculated by Rule II.; or the months and days may be reduced to days, and the interest for them calculated by Rule III. In the selection of the methods to be practised, the computist must be guided by the peculiar circumstances of the question.

Example 4. — Find the interest of 462l. for 85 days, at 5 per cent. by Rule III.

$$x = \frac{462 \times 5 \times 85}{36500} = \text{£}5 \ 7 \ 7$$

Example 5. — What is the amount of interest on 75l. 8s. 6d. at $4\frac{1}{2}$ per cent. for 4 years, 7 months, and 27 days?

ANALYSIS. — Since the calendar months have unequal lengths, it would be necessary, in order to make this computation, to know the time of the year at which the interest begins to accrue; but we shall, in the present case, consider each month as a twelfth part of the year, and make the computation by the rules already given. The principal, reduced to decimals of a pound, is 75·425l.: the rate expressed in decimals is 4·5; we have, then, —

	£	s.	d.
Interest for 4 years	$\left. \begin{array}{l} \text{Interest for 4} \\ \text{years} \end{array} \right\} = \frac{75\cdot425 \times 4\cdot5 \times 4}{100} = 13\cdot5765 = 13 \ 11 \ 6\frac{1}{2}$		
Interest for 7 months	$\left. \begin{array}{l} \text{Interest for 7} \\ \text{months} \end{array} \right\} = \frac{75\cdot425 \times 4\cdot5 \times 7}{1200} = 7\cdot97999 = 1 \ 19 \ 7\frac{1}{2}$		
Interest for 27 days	$\left. \begin{array}{l} \text{Interest for 27} \\ \text{days} \end{array} \right\} = \frac{75\cdot425 \times 4\cdot5 \times 27}{36500} = 0\cdot251 = 0 \ 5 \ 0\frac{1}{4}$		
	$\text{£}15 \ 16 \ 2\frac{1}{4}$		

Calculations of interest are, however, made with much

greater expedition and accuracy by the aid of tables of interest, which tables themselves, however, have been previously calculated by the methods just explained. We shall not here enter into further details than to explain the method of using such tables. The following table exhibits the interest on 1*l*., in decimals of a pound, for any number of years up to a certain limit, at the several rates of 3 per cent., $3\frac{1}{2}$ per cent., 4 per cent., $4\frac{1}{2}$ per cent., and 5 per cent.

TABLE I. — A TABLE OF SIMPLE INTEREST.

THE INTEREST OF ONE POUND FOR A NUMBER OF YEARS.

Years.	At 3 per cent.	$3\frac{1}{2}$ per cent	4 per cent.	$4\frac{1}{2}$ per cent.	5 per cent	Years
1	·03	·035	·04	·045	·05	1
2	·06	·07	·08	·09	·1	2
3	·09	·105	·12	·135	·15	3
4	·12	·14	·16	·18	·2	4
5	·15	·175	2	·225	·25	5
6	·18	·21	·24	·27	·3	6
7	·21	·245	·28	·315	·35	7
8	·24	·28	·32	·36	·4	8
9	·27	·315	·36	·405	·45	9
10	·3	·35	·4	·45	·5	10
11	·33	·385	·44	·495	·55	11
12	·36	·42	·48	·54	·6	12
13	·39	·455	·52	·585	·65	13
14	·42	·49	·56	·63	·7	14
15	·45	·525	·6	·675	·75	15
16	·48	·56	·64	·72	·8	16
17	·51	·595	·68	·765	·85	17
18	·54	·63	·72	·81	·9	18
19	·57	·665	·76	·855	·95	19
20	·6	·7	·8	·9	1·	20
21	·63	·735	·84	·945	1·05	21
22	·66	·77	·88	·99	1·1	22
23	·69	·805	·92	1·035	1·15	23
24	·72	·84	·96	1·08	1·2	24
25	·75	·875	1·	1·125	1·25	25

The following is part of a table which gives the interest on £1., in like manner, for any number of days up to a certain limit.

TABLE II. — A TABLE OF SIMPLE INTEREST.

THE INTEREST OF ONE POUND FOR ANY NUMBER OF DAYS, ETC.

Days	3 per cent.	3½ per cent.	4 per cent.	4½ per cent.	5 per cent.	Days
1	·0000,821	·0000,958	·0001,095	·0001,232	·0001,369	1
2	·0001,641	·0001,916	·0002,191	·0002,465	·0002,739	2
3	·0002,465	·0002,876	·0003,287	·0003,698	·0004,109	3
4	·0003,287	·0003,835	·0004,383	·0004,931	·0005,479	4
5	·0004,109	·0004,794	·0005,479	·0006,164	·0006,849	5
6	·0004,931	·0005,753	·0006,575	·0007,397	·0008,219	6
7	·0005,753	·0006,712	·0007,671	·0008,630	·0009,589	7
8	·0006,575	·0007,671	·0008,767	·0009,863	·0010,958	8
9	·0007,397	·0008,630	·0009,863	·0011,095	·0012,328	9
10	·0008,219	·0009,589	·0010,958	·0012,328	·0013,698	10
11	·0009,041	·0010,547	·0012,054	·0013,561	·0015,068	11
12	·0009,863	·0011,506	·0013,150	·0014,794	·0016,438	12
13	·0010,684	·0012,465	·0014,246	·0016,027	·0017,808	13
14	·0011,506	·0013,424	·0015,342	·0017,260	·0019,178	14
15	·0012,328	·0014,383	·0016,438	·0018,493	·0020,547	15
16	·0013,150	·0015,342	·0017,530	·0019,726	·0021,917	16
17	·0013,972	·0016,301	·0018,630	·0020,958	·0023,287	17
18	·0014,794	·0017,260	·0019,726	·0022,191	·0024,657	18
19	·0015,616	·0018,219	·0020,821	·0023,424	·0026,027	19
20	·0016,438	·0019,178	·0021,917	·0024,657	·0027,397	20
21	·0017,260	·0020,137	·0023,013	·0025,890	·0028,767	21
22	·0018,082	·0021,095	·0024,109	·0027,123	·0030,137	22
23	·0018,904	·0022,054	·0025,205	·0028,356	·0031,506	23
24	·0019,726	·0023,013	·0026,301	·0029,580	·0032,876	24
25	·0020,547	·0023,972	·0027,397	·0030,821	·0034,246	25
26	·0021,369	·0024,931	·0028,493	·0032,054	·0035,616	26
27	·0022,191	·0025,890	·0029,589	·0033,287	·0036,986	27
28	·0023,013	·0026,849	·0030,684	·0034,520	·0038,356	28
29	·0023,835	·0027,808	·0031,780	·0035,753	·0039,726	29
30	·0024,657	·0028,767	·0032,876	·0036,986	·0041,095	30

When the interest of any sum of money is required at a given rate, for any given number of years, look in the first table for the given number of years in the first or last column, and, in the same horizontal line with that number, you will find under the given rate of interest the decimal of a pound, which expresses the interest of 1*l.* for the required number of years. If this decimal be multiplied by the principal, the product will be the interest required.

*Example 6. — What is the interest on 38*l.*, at 3½ per cent., for 17 years?*

In Table I., opposite 17, and under the column of 3½ per cent., we find the decimal .595, which is the interest upon 1*l.* for 17 years: we multiply this by 38, and the product is 22.61*l.* = 22*l.* 12*s.* 2½*d.*

*Example 7. — To find the interest on 5*l.* 12*s.* 6*d.*, at 4 per cent., for 23 years and 6 months.*

In Table I., opposite 23 years, and in the column under 4 per cent., we find the decimal .92, which is the interest on 1*l.* for 23 years: opposite 1 year we find the decimal .04, which is the interest upon 1*l.* for 1 year; half of the latter, which is 0.02, is, therefore, the interest on 1*l.* for 6 months; hence, the interest on 1*l.* for 23 years and 6 months is .94. Multiplying this by the principal, which reduced to decimals, is 5.625, we obtain the product 5.3875*l.* = 5*l.* 7*s.* 9½*d.*, which is the interest required.

*Example 8. — Find the interest on 75*l.* 8*s.* 6*d.* for 4 years, 7 months, and 27 days, at 4½ per cent.*

In the Table I. above, opposite 4 years, in the column under 4½ per cent., we find the decimal .18, which is the interest upon 1*l.* for 4 years: we shall find the interest on 1*l.* for 6 months by taking half the interest on 1*l.* for 1 year; this appears, by the same table, to be .045, which, divided by 2, gives .0225, which is, therefore, the interest on 1*l.* for 6 months: the sixth part of this is .00375, which is the interest for 1 month; adding this to the interest for 6 months, we find the interest for 7 months to be .02625. To find the in-

terest for 27 days we refer to Table II., and opposite 27 days, and under the column headed $4\frac{1}{2}$ per cent., we find the decimal $\cdot 0033287$. Add together the amounts of interest thus found for 4 years, 7 months, and 27 days.

Interest for 4 years	=	$\cdot 18$
Interest for 7 months	=	$\cdot 02625$
Interest for 27 days	=	$\cdot 0033287$
<hr/>		
Total interest on £1	=	$\cdot 2095787$

Multiplying this interest by the principal $75\cdot 425$, and omitting all the digits after the fourth decimal place in the product, we obtain $15\cdot 8071 = 15\text{ l. } 16\text{ s. } 2\text{ d.}$, which is the interest sought.

(416.) If the lender instead of receiving from the borrower the interest accruing due upon his principal from year to year, leaves that interest in the hands of the borrower, it may be regarded as so much added to the principal, at the times at which it falls due. Thus, if 100 l. be the sum lent, and the interest be payable yearly at 5 per cent., then at the end of the first year the principal will become 105 l., and the interest at the end of the second year will therefore be, not the interest on 100 l., but the interest on 105 l. In this manner the principal which produces the interest each year, is increased by the amount of the interest of the preceding year.

When interest is thus chargeable upon interest, the increase upon the original principal is called **COMPOUND INTEREST**.

To find the interest on any principal invested at compound interest for any number of years, it is necessary to find the amount of the principal and interest at the end of each year, and considering this as a new principal, compute the interest upon it for the next year.

Example 9. — To find the compound interest for 5 years on 120 l. at 5 per cent., the calculation would be as follows : —

First year's principal	-	£120	0	0
First year's interest	-	6	0	0
Second year's principal	-	126	0	0
Second year's interest	-	6	6	0
Third year's principal	-	132	6	0
Third year's interest	-	6	12	3½
Fourth year's principal	-	138	18	3½
Fourth year's interest	-	6	18	10½
Fifth year's principal	-	145	17	2
Fifth year's interest	-	7	5	10
Amount at the end of } fifth year -		£153	3	0

Example 10. — To find the time in which a principal sum will be doubled at Compound Interest at 5 per cent.

Let us suppose the principal to be 100*l.* ; the computation must be continued until a principal be obtained amounting to 200*l.*

First year's principal	-	£100	0	0
First year's interest	-	5	0	0
Second year's principal	-	105	0	0
Second year's interest	-	5	5	0
Third year's principal	-	110	5	0
Third year's interest	-	5	10	3
Fourth year's principal	-	115	15	3
Fourth year's interest	-	5	15	9
Fifth year's principal	-	121	11	0
Fifth year's interest	-	6	1	7
Sixth year's principal	-	127	12	7
Sixth year's interest	-	6	7	8
Seventh year's principal	-	134	0	3
Seventh's year's interest	-	6	14	0
Eighth year's principal	-	140	14	3
Eighth year's interest	-	7	0	9
Ninth year's principal	-	147	15	0
Ninth year's interest	-	7	7	9

Tenth year's principal	-	155	2	9
Tenth year's interest	-	7	15	2
Eleventh year's principal	-	162	17	11
Eleventh year's interest	-	8	2	11
Twelfth year's principal	-	171	0	10
Twelfth year's interest	-	8	11	0
Thirteenth year's principal		179	11	10
Thirteenth year's interest	-	8	19	6
Fourteenth year's principal		188	11	4
Fourteenth year's interest	-	9	8	6
Fifteenth year's principal	-	197	19	10
Fifteenth year's interest	-	9	19	0
Amount at the end of 15 years	- - - }	£207	18	10

From this calculation it appears that the principal will be more than doubled in 15 years. If it be required to find the period in which it will be exactly doubled, we have only to find the number of days which 197*l.* 19*s.* 10*d.* must be placed at interest to produce 2*l.* 0*s.* 2*d.*, or, what will be nearly the same, how many days 198*l.* must be placed at interest to produce 2*l.* To solve this question we would have the following statement, transposing the effects as usual, —

Agents	-	100	-	198
Effects	-	2	-	5
Days	-	365	-	<i>x.</i>

Multiplying the numbers in the first column we obtain the product 73000, and dividing this by the product of the first two numbers in the second column we obtain the quotient 74. The principal would, therefore, be doubled in 14 years and 74 days, at compound interest. In the above calculation small fractions have been omitted.

When the principal is large and the time considerable, the computations at compound interest are generally complex and embarrassing. By the aid of the higher mathematics and logarithmic tables more compendious

methods may be obtained ; but, the most convenient and expeditious means of calculation are tables of compound interest, by which all such problems may be solved with very little or no calculation.

DISCOUNT.

(417.) When a payment is made at any time previous to the date at which it falls due, the party thus anticipating the payment is entitled to deduct from the amount to be paid a certain sum, in consideration of the interest which would accrue on the money paid, between the time of the actual payment and the time at which the amount would fall due. This deduction or abatement is called *Discount*.

Thus, if it be required to ascertain the present value of a bill of exchange for a certain amount, which will fall due at a future period, it will be necessary to deduct from the actual amount for which the bill is drawn the amount of discount.

Again, if it be proposed to pay a tradesman in cash, for goods for which it is the custom of the trade to give a certain length of credit, the buyer is entitled to discount, proportionate to the amount of the account and the length of credit.

The method of computing discount will be easily deduced from an example.

Example 1.—A merchant presents a bill of exchange for 3000*l.* payable at the end of one year to his banker to be discounted. It is required to know what discount the banker should deduct from the amount of the bill?

ANALYSIS. — It is evident that the question here to be determined is, what is that sum of money which being now placed at interest would, at the end of one year, be worth 3000*l.*? for such is the sum which the banker ought to pay to the merchant ; and the difference between this sum, whatever it be, and 3000*l.* is the amount of the discount. To solve this question, it is therefore necessary that the rate of interest should be

previously settled: suppose this to be 6 per cent. A present sum of 100*l.* would, at this rate, increase to the amount of 106*l.* at the end of one year: the question, therefore, is, if 100*l.* at the end of one year become 106*l.* what sum, at the end of one year, would become 3000*l.*? The statement would be thus:—

$$£106 : £3000 = £100 : x.$$

COMPUTATION. — To find the amount represented by *x*, multiply the amount of the bill by 100, and divide the product by 106: the quotient will, in this case, be 2830*l.* 3*s.* 9½*d.*, which is the sum which the banker should pay to the merchant for the bill; the difference between this and 3000*l.*, which is 169*l.* 16*s.* 2½*d.*, gives the amount of the discount.

The above analysis has led us to compute the sum to be paid by the merchant to the banker; but it is more convenient and usual to compute, in the first instance the amount of the discount, which is easily done.

ANALYSIS. — If a bill for 106*l.* was presented to the banker, it is clear that the sum which he would be entitled to deduct from it for discount would be 6*l.*, inasmuch as 100*l.* paid at present, would, at the end of a year, be worth 106*l.* The question therefore is, if 106*l.* gives a discount of 6*l.* for one year, what discount will be given for one year by 3000*l.*? which leads to the following statement:—

$$£106 : 6 = £3000 : x.$$

where *x* expresses the discount required.

COMPUTATION. — To find *x*, multiply 3000 by 6, and divide the product by 106: the quotient will be 169*l.* 16*s.* 2½*d.*, which is the discount sought.

In the preceding example, the time for which the discount is computed is one year, a circumstance which renders the analysis more simple than if any other time had been supposed. We shall now take a more general case.

Example 2. — To find the discount to be deducted

for the present payment of 4850*l.*, payable in 13 years 6 months, the rate of interest being 5 per cent. per annum.

ANALYSIS. — In this case, the sum which would produce a discount of 5 per cent. in one year would be 105*l.*: the statement will then be as follows:—

Agents	-	105	-	4850
Effects	-	5	-	<i>x</i>
Time	-	1 $\frac{1}{2}$	-	13 $\frac{1}{2}$ 6 ^m .

Transposing the effects and reducing the amounts to decimals of a year, the statement becomes—

Agents	-	105	-	4850
Effects	-	<i>x</i>	-	5
Time	-	1 $\frac{1}{2}$	-	13.5 $\frac{1}{2}$.

Multiply the three numbers in the second column, and find their continued product; divide this by 105, and the quotient will be 3117.952*l.* = 3117*l.* 19*s.* 0 $\frac{1}{2}$ *d.* which is the discount sought. In this process we have multiplied the principal by the rate of interest and multiplied the product thus obtained by the time in years, and decimals of a year. The product thus found is then divided by 105, which is the number found by adding the rate of interest to 100: hence we obtain the following

RULE.

(418.) *To find the discount to be allowed for the present payment of a given sum, due at a future time, multiply the given sum by the rate of interest, and multiply the product thus obtained by the time from the present date until the date at which the sum falls due, this time being expressed in years and decimals of a year: divide the continued product thus obtained by the number which you will find by adding the rate of interest to 100; the quotient will be the discount sought.*

It is sometimes more convenient to express the time in months or days, than in decimals of a year.

Example 3. — *To find the discount on a bill for 25*l.* 12*s.* 6*d.* due in 1 year and 5 months from the present time, the rate of interest being 5 per cent.*

ANALYSIS. — The statement will be as follows, reducing the time to months :—

Agents	-	105	-	£25 12 6
Effects	-	5	-	<i>x</i>
Months	-	12	-	17.

Transposing the effects, and converting the shillings and pence into decimals of a pound, the statement becomes—

Agents	-	105	-	25.625
Effects	-	<i>x</i>	-	5
Months	-	12	-	17.

Multiplying the numbers in the second column, and dividing their continued product by the product of 105 and 12 in the first column, we obtain the quotient £1.728=1*l.* 14*s.* 6½*d.*, which is the discount sought.

From this example we derive the following

RULE.

(419.) *If the time be expressed in months, multiply the amount to be paid by the rate of interest, and multiply the product by the number of months from the present time to the time it will be due : divide the product thus found by the product which is found by multiplying by 12 the number obtained by adding the rate of interest to 100.*

If the time be expressed in days, there will be no other difference in the statement, except that the third term in the first column will then be 365, and we shall have the following

RULE.

(420.) *When the time is expressed in days, multiply as before, the amount to be discounted by the rate of interest, and the product by the number of days ; the product thus found must then be divided by another product, which will be found by multiplying 365 by the number obtained by adding the rate of interest to 100.*

The above are the fair principles upon which all dis-

count should be computed. It has been, however, customary in this country, in commercial business, instead of considering discount under this point of view, to compute it merely as the interest which the whole amount to be discounted would produce, from the present time to the time at which the amount falls due. Thus the discount on a bill for 100*l.*, payable in a year, at 5 per cent. interest, would be considered to be 5*l.*, whereas 5*l.* is, in truth, the discount on a bill for 105*l.* payable in a year.

According to this established custom, discount would be calculated by the following

RULE.

(421.) *Find the interest on the sum to be discounted, from the day on which it is discounted to the day on which it becomes payable, and this interest will be the discount.*

PROFIT AND LOSS.

(422.) When money is placed at interest it is always supposed that the principal is secured to its owner, and that he enters into no speculation upon it. When a sum of money is invested in any speculation, it is called **CAPITAL**, and the increase which it receives from year to year is called **PROFIT**. Profit, like interest, is calculated at so much on every 100*l.* in a year, or so much per cent. per annum, supposing the profit from year to year to be uniform. The same methods of calculation will be applicable to profit on capital as have been already explained for computing interest on principal.

To compute the rate of profit, a merchant will estimate his capital and stock at the beginning and end of the year, and he will subtract the amount at the beginning from the amount at the end; the remainder will be the total profit. To find the profit per cent. it will be only necessary to annex two ciphers to the total profit, and divide the number thus obtained by the capital at the beginning of the year.

BROKERAGE, COMMISSION, &c.

(423.) **BROKERAGE, COMMISSION, &c.** are allowances of so much per cent. which are usually made to factors or agents, employed by merchants to effect sales. In such computations time does not enter as an element, and the problem is merely confined to determine a given fraction of a sum of money.

*Example 1. — To find the brokerage or commission on sales amounting to 539*l.* 14*s.* at 2½ per cent.*

The question here to be solved is, if 100*l.* produce a commission of 2*l.* 10*s.*, what commission will 539*l.* 14*s.* produce? We have hence the following proportion:—

$$£100 : £539\ 14\ 0 = £2\ 10 : x.$$

where x expresses the commission sought; we therefore must multiply 539*l.* 14*s.* by 2*l.* 10*s.*, and divide the product by 100. The quotient will in this case be 13*l.* 9*s.* 10¾*d.* From this analysis we derive the following

RULE.

(424.) *To find the total commission, at a given rate per cent., on a given amount, multiply the amount by the rate, and divide the product by 100: the quotient will be the total commission sought.*

It frequently happens that it is more convenient to express the rate of commission by so much in the pound, than by so much in the 100*l.* Thus a commission of 5 per cent. will be a shilling in the pound, and in such case the total commission will always be found by taking the number of pounds in the amount to express shillings. Again, if the commission be 2½ per cent., it will be equivalent to 6*d.* in the pound, and we shall therefore find the commission in shillings, by dividing the number of pounds by 2.

INSURANCE.

(425.) **INSURANCE** is an indemnity given to make good a future contingent loss, in consideration of a per centage paid down on the extent of the value insured. The party insuring thus undertakes to make good any

loss of property incurred by fire, by storms, or other accidents at sea, by loss of life, &c. &c.

The per centage on the amount insured is payable yearly, and is called the *premium*. Its amount is calculated by the rule already explained for computing commission, &c.

TARE AND TRET.

(426.) The **GROSS WEIGHT** of any sort of merchandise is the weight which is found by weighing it with the box, sack, barrel, or whatever be the envelope in which it shall be contained.

TARE is an allowance made to the buyer, for the weight of the envelope containing the merchandise.

TRET is an allowance made to the buyer for waste.

When these, or other similar allowances are deducted from the gross weight, the remainder is called *net weight*. Generally, such allowances are either a fraction of the total weight, or a given fraction of the value of the merchandise. In any case, the methods of computation consist in finding the required fraction of the whole weight or value of the merchandise.

When such allowances are expressed at so much per parcel, their total amount will be found by multiplying the allowance by the number of parcels.

When they are expressed at so much per hundred weight, we have only to find what fractional part of a hundred weight the allowance is, and multiply the total weight by the fraction thus obtained: the product will be the total amount of the allowance.

Example 1. — *A broker is employed to sell goods to the amount of 627l. 10s. on an allowance of 2s. 8d. per cent. : what is his total brokerage?*

ANALYSIS. — Divide the total amount by 100, and multiply the quotient by 2s. 8d.

COMPUTATION. — Dividing 627l. 10s. by 100, we get the quotient 6·275, and multiplying 2s. 8d. by 6·275, we obtain the product 16s. 8 $\frac{3}{4}$ d., which is the amount of the brokerage sought.

Example 2. — A factor purchases goods to the amount of 500*l.* 14*s.* at a commission of $2\frac{1}{2}$ per cent., what is the total amount of his commission?

COMPUTATION. — Divide 500*l.* 14*s.* by 100, and we obtain the quotient 5*l.* 0*s.* 1 $\frac{6}{100}$ *d.*: multiply this by 2*l.* 10*s.* and we obtain the product 12*l.* 10*s.* 4 $\frac{1}{2}$ *d.*

Example 3. — A factor sells goods to the amount of 230*l.* 12*s.* at a commission of 3 per cent.: what is the total amount of his commission?

COMPUTATION. — Divide 230*l.* 12*s.* by 100, and the quotient is 2*l.* 6*s.* 1 $\frac{1}{100}$ *d.*: multiply this by 3, and the product is 6*l.* 18*s.* 4 $\frac{3}{100}$ *d.*, which is the commission sought.

Example 4. — What premium must be paid for insuring goods to the amount of 317*l.* 18*s.* 6*d.* at $1\frac{3}{4}$ per cent.

COMPUTATION. — Divide 317*l.* 18*s.* 6*d.* by 100, and the quotient is 3*l.* 3*s.* 7 $\frac{1}{100}$ *d.*: multiply this by 1*l.* 15*s.* and the product will be 5*l.* 11*s.* 3 $\frac{2}{100}$ *d.*

PARTNERSHIP.

(427.) When two or more persons subscribe to a common-stock as capital in trade, and carry on a joint business, the profits which accrue from year to year on the common capital should be distributed between the partners, in the proportion of the capital they respectively subscribe, provided that the capital of each partner remains for the same time invested in the business.

Let us suppose that the whole joint stock in trade consists of a number of equal shares, represented each by 1*l.* of the original capital. It is clear that, in the distribution of profits, the total profits should be divided into as many equal parts as there are pounds in the common capital subscribed; and that, in the distribution of profits, each partner should receive as many of those parts as there were pounds in his subscribed capital. This will be easily understood by an example.

Example 1. — Let A, B, and C, invest respectively 105*l.* 75*l.* and 55*l.* in trade, and at the end of the first year suppose that a profit of 468*l.* 15*s.* has been ob-

tained. *It is required to know how this profit must be distributed between the partners?*

Adding together the portions of capital subscribed, we find the total capital invested to be 2368*l*. Let us consider this, the whole stock in trade, to consist of 2368 shares of 1*l*. each. It is evident, therefore, that if the whole profit be divided by 2368, we shall obtain the profit on each share of 1*l*. To find therefore the portion of profit to which each partner is entitled, we have only to multiply the profit on 1*l*. by the number of pounds in his subscribed capital: the process would then be as follows:—

Divide the total profit by the number of pounds in the total capital, and multiply the quotient by the number of pounds in the subscribed capital: the product will be the portion of the total profit to which each partner is entitled. We should obtain the same result by reversing the order of these two operations, viz. by first multiplying the total profit by the number of pounds in the subscribed capital of each partner, and then dividing the product by the number of pounds in the total subscribed capital. This method would, in practice, generally be the most expeditious. In the present example the process of calculation would be as follows:—

			£	s.	d.
Total profit	-	-	468	15	0
x A's capital	-	-	1055	0	0
Divide by total capital	-	2368	494231	5	0
A's share of profit	-		208	16	$9\frac{43}{592}$
Total profit	-	-	468	15	0
x B's capital	-	-	756	0	0
Divide by total capital	-	2368	954375	0	0
B's share of profit	-		149	13	$0\frac{228}{592}$
Total profit	-	-	468	15	0
x C's capital	-	-	557	0	0
Divide by total capital	-	2368	261093	15	0
C's share of profit	-		£ 110	5	$2\frac{13}{592}$

To verify this computation, we need only add together the three shares into which the whole profit is to be divided ; and if the total obtained by this addition amount to 468*l.* 15*s.* the computation has been correctly performed.

		£	s.	d.
A's share of profit	-	208	16	9 ²⁴³ ₅₉₂
B's share of profit	-	149	13	0 ²²⁸ ₅₉₂
C's share of profit	-	110	5	2 ²¹ ₅₉₂
		<u>£468 15 0</u>		

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